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Two New Solutions for an Inverse Problem on the Length of the Meridian Arc

Der Beitrag bietet zwei neue Lösungen zur Berechnung der geographischen Breite aus einer Meridianbogenlänge. Eine Lösung wird direkt aus einer Taylorreihenentwicklung abgeleitet, die zweite Lösung verwendet die Hermite-Interpolation.

1 Introduction

The length of the meridian arc is one fundamental quantity in ellipsoid geodesy and Gauss-Krueger projection. It is an integral of ellipse and consequently is not analytically integrable. In practice, it is usually expanded into a series of the eccentricity in order to integrate it term by term (KLOTZ, 1991). This series converges very quickly since the eccentricity of a reference ellipsoid is very small ($e^2 \approx 1/150$).

The procedure from the known length of the meridian arc to its corresponding latitude, or an inverse problem on the meridian arc, is a bit troublesome and therefore there exist various different solutions. In general, these solutions might be divided into the iterative ones and the direct ones. The iterative solutions are namely to use the formula for the length of the meridian arc and solve the corresponding latitude by Newton iteration method. The direct solutions are to express the corresponding latitude in terms of a trigonometric series (MITTERMAYER, 1965; ZHOU, 1984) or a polynomial of the meridian length (ZHU, 1978).

An inverse problem on the length of the meridian arc seems to be a solved and out-of-fashion one. Examining the existing inverse solutions with a slight care, however, one will find that this problem has not been solved perfectly and ideally. This situation, of course, is due to the complexity of this problem itself (an inverse problem of ellipse function). On the other hand, it is also because a few researchers did not grasp the essential point of the problem.

This paper reconsiders an inverse problem on the length of the meridian arc. Making use of the derivative rule of implicit function as well as the Hermite interpolation principle, two series solutions for an inverse problem on the length of the meridian arc are presented

2 A Power Series Solution for an Inverse Problem on the Length of the Meridian Arc

From (XONG, 1988), the derivative of the meridian arc with respect to geodetic latitude B reads

$$\frac{dX}{dB} = a(1-e^2)/(1-e^2\sin^2 B)^{3/2} \tag{1}$$

where B is geodetic latitude, a is the semi-major axis of a reference ellipsoid, e^2 is the eccentricity square and X is the length of the meridian arc from an equator to a point with geodetic latitude B.

For convenience, we introduce two new denotations

$$\overline{w} = \frac{2}{\pi}\int_0^{\pi/2} dB/(1-e^2\sin^2 B)^{3/2} = 1+\frac{3}{4}e^2+\frac{45}{64}e^4+\frac{175}{256}e^6+\cdots \tag{2}$$

$$x = X/a(1-e^2)\overline{w} \tag{3}$$

It is easy to prove that a new variable corresponds $x = 0$ and $x = \pi/2$ when $X = 0$ and $X = a(1-e^2)\,\overline{w}\,\pi/2$ respectively. Differentiating eq. (3) and inserting it into eq. (1) result in

$$dx = dX/a(1-e^2)\overline{w} = dB/(1-e^2\sin^2 B)^{3/2}\overline{w}$$

or

$$\frac{dB}{dx} = \overline{w}(1-e^2\sin^2 B)^{3/2} \tag{4}$$

Eq. (4) is an implicit function of x. Therefore, it seems difficult to expand it into a power series of $\sin x$ .To expand eq. (4) into a power series of $\sin x$, we introduce a new variable

$$t = e^2\sin^2 x \tag{5}$$

and denote eq. (4) as

$$f(t) = \frac{dB}{dx} \tag{6}$$

Now we try to expand $f(t)$ into the following power series

$$f(t) = \frac{dB}{dx} = f_t(0) + f_t'(0)t + \frac{1}{2!}f_t''(0)t^2 + \frac{1}{3!}f_t'''(0)t^3 + \frac{1}{4!}f_t^{(4)}(0)t^4 + \cdots \tag{7}$$

where $f_t(0), f_t^n(0)\cdots\cdot$ are the corresponding derivatives of $f(t)$ with respect to t respectively.

According to the definition of eq. (4–6), it holds

$$f_t(0) = \overline{w} \tag{8}$$

$$dt = 2e^2\sin x\cos x dx \Rightarrow \frac{dx}{dt} = 1/e^2\sin 2x \tag{9}$$

The derivatives of $f(t)$ can be in turn determined by the chain rule for implicit functions. As a result, we have

$$f_t'(t)=\frac{df}{dB}\frac{dB}{dx}\frac{dx}{dt}=-\frac{3}{2}\overline{w}(1-e^2\sin^2 B)^{1/2}e^2\sin 2B\overline{w}(1-e^2\sin^2 B)^{3/2}\frac{1}{e^2\sin 2x}$$

When $t\to 0$, $x\to 0$, $B\to 0$, it is easy to know

$$\lim_{t\to 0}\frac{\sin 2B}{\sin 2x}=\lim_{x\to 0}\frac{B}{x}=\lim_{x\to 0}\frac{\Delta B}{\Delta x}=\lim_{x\to 0}\frac{dB}{dx}=\overline{w}(1-e^2\sin^2 B)^{3/2}$$

Therefore, we get

$$\lim_{t\to 0}f_t'(t)=-\frac{3}{2}\overline{w}^3(1-e^2\sin^2 B)^{7/2} \tag{10}$$

It is difficult to determine the strict derivatives of $f(t)$ with the order higher than two. Under the assumption that the order of limit and derivative is exchangeable, we approximately have

$$\lim_{t\to 0}f_t''(t)=\frac{df'(t)}{dB}\frac{dB}{dx}\frac{dx}{dt}=\frac{3}{2}\cdot\frac{7}{2}\overline{w}^5(1-e^2\sin^2 B)^{11/2} \tag{11}$$

$$\lim_{t\to 0}f_t'''(t)=\frac{df''(t)}{dB}\frac{dB}{dx}\frac{dx}{dt}=-\frac{3}{2}\cdot\frac{7}{2}\cdot\frac{11}{2}\overline{w}^7(1-e^2\sin^2 B)^{15/2} \tag{12}$$

$$\lim_{t\to 0}f^{(4)}(t)=\frac{df'''(t)}{dB}\frac{dB}{dx}\frac{dx}{dt}=\frac{3}{2}\cdot\frac{7}{2}\cdot\frac{11}{2}\cdot\frac{15}{2}\overline{w}^9(1-e^2\sin^2 B)^{19/2} \tag{13}$$

……

$$\lim_{t\to 0}f^{(n)}(t)=(-1)^n\frac{3\cdot 7\cdots(4n-1)}{2^n}\overline{w}^{2n+1}(1-e^2\sin^2 B)^{\frac{4n+3}{2}} \tag{14}$$

Taking into account $B=0$ when $t=0$, we arrive at

$$f_t'(0)=-\frac{3}{2}\overline{w}^3 \tag{15}$$

$$f_t''(0)=\frac{21}{4}\overline{w}^5 \tag{16}$$

$$f_t'''(0)=-\frac{3\cdot 7\cdot 11\cdot}{8}\overline{w}^7 \tag{17}$$

$$f_t''''(0)=\frac{3\cdot 7\cdot 11\cdot 15}{16}\overline{w}^9 \tag{18}$$

……

$$f_t^{(n)}(0)=(-1)^n\frac{3\cdot 7\cdots\cdots(4n-1)}{2^n}\overline{w}^{2n+1} \tag{19}$$

Inserting eq. (8) and eqs. (15–18) into eq. (7), we get the power series of $f(t)$ with respect to $e\sin x$

$$\begin{aligned}f(t)=\overline{w}&-\frac{3}{2}\overline{w}^3e^2\sin^2 x+\frac{1}{2}\cdot\frac{3\cdot 7}{4}\overline{w}^5e^4\sin^4 x\\&-\frac{1}{6}\cdot\frac{3\cdot 7\cdot 11}{8}\overline{w}^7e^6\sin^6 x+\frac{1}{24}\cdot\frac{3\cdot 7\cdot 11\cdot 15}{16}\overline{w}^9e^8\sin^8 x\end{aligned} \tag{20}$$

Further, making use of the following trigonometric identities

$$\sin^2 x=\frac{1}{2}-\frac{1}{2}\cos 2x$$

$$\sin^4 x=\frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$$

$$\sin^6 x=\frac{5}{16}-\frac{1}{32}\cos 2x+\frac{3}{16}\cos 4x-\frac{1}{32}\cos 6x$$

$$\sin^8 x=\frac{35}{128}-\frac{7}{16}\cos 2x+\frac{7}{32}\cos 4x-\frac{1}{16}\cos 6x+\frac{1}{128}\cos 8x$$

the integration of eq. (20) with respect to x finally yields a trigonometric series solution for an inverse problem on the length of the meridian arc as

$$\begin{aligned}B(x)=\int_0^x f(t)dx=x&+(\frac{3}{8}e^2-\frac{21}{32}e^4\overline{w}^2+\frac{1155}{1024}e^6\overline{w}^4-\frac{8085}{4096}e^8\overline{w}^6)\overline{w}^3\sin 2x\\&+(\frac{21}{256}e^4-\frac{231}{1024}e^6\overline{w}^2+\frac{8085}{16384}e^8\overline{w}^4)\overline{w}^5\sin 4x\\&+(\frac{77}{3072}e^6-\frac{385}{4096}e^8\overline{w}^2)\overline{w}^7\sin 6x+\frac{1155}{131072}e^8\overline{w}^9\sin 8x\end{aligned} \tag{21}$$

For convenience in application, inserting the expression of $\overline{w}$ into eq. (21), eq. (21) can be rearranged as (as this procedure is tedious, one could do it by means of MATHEMATICA alternatively. MATHEMATICA is one of the most popular symbolic computation softwares)

$$\begin{aligned}B(x)=x&+(\frac{3}{8}e^2+\frac{3}{16}e^4+\frac{93}{1024}e^6+\frac{129}{2048}e^8)\sin 2x\\&+(\frac{21}{256}e^4+\frac{21}{256}e^6+\frac{483}{8192}e^8)\sin 4x\\&+(\frac{77}{3072}e^6+\frac{77}{2048}e^8)\sin 6x+\frac{115}{131072}e^8\sin 8x\end{aligned} \tag{22}$$

It is worthy to point out that the coefficient of x in eq. (21–22) should theoretically be one. In effect, the linear coefficient in the derivation above is equal to

$$\overline{w}-\frac{3}{4}\overline{w}^3+\frac{1}{2}\frac{21}{4}\frac{3}{8}e^4\overline{w}^5-\frac{1}{6}\frac{3\cdot 7\cdot 11}{8}\frac{5}{16}e^6\overline{w}^7+\frac{1}{24}\frac{3\cdot 7\cdot 11\cdot 15}{16}\frac{35}{128}e^8\overline{w}^9=1+\frac{6}{256}e^6+\frac{246}{16384}e^8$$

Its error is up to the order of e^6. This error, we guess, might be caused by exchanging the order of limit and derivative, or by the software MATHEMATICA itself. Anyway, the coefficient of x has to be set to one.

3 An Interpolation Solution for an Inverse Problem on the Length of the Meridian Arc

In the section 2, we formulated an analytical solution for an inverse problem on the length of the meridian arc. Its correctness and accuracy, however, leave to be proven and checked. It is not sufficient reasonable to exchange the order of limit and derivative.

As an alternative, an interpolation method is applied to an inverse problem on the length of the meridian arc in this section.

Suppose an interpolation solution for an inverse problem on the length of the meridian arc as

$$B(x)=x+a_2\sin 2x+a_4\sin 4x+a_6\sin 6x+a_8\sin 8x \tag{23}$$

where a_{2i} is the coefficients to be determined.

We will determine these coefficients by making use of function values and derivative values at some specific points, namely a Hermite interpolation.

First from the following formula (Xong, 1988)

$$
\begin{cases}
x = B - (\alpha \sin 2B - \beta \sin 4B + \gamma \sin 6B + \cdots)/\overline{w} \\
\alpha = \frac{3}{8}e^2 + \frac{15}{32}e^4 + \frac{525}{1024}e^6 + \frac{2205}{16384}e^8 \\
\beta = \frac{15}{256}e^4 + \frac{105}{1024}e^6 + \frac{2205}{16384}e^8 \\
\gamma = \frac{35}{3072}e^6 + \frac{105}{4096}e^8
\end{cases}
\tag{24}
$$

$B(x)$ at $x = \pi/4$ can be with iteration determined. Let $B(\pi/4) \approx \pi/4$ as an initial value, we approximately have

$$B(\frac{\pi}{4}) \approx x + (\alpha \sin\frac{\pi}{2} - \beta \sin \pi + \gamma \sin\frac{3\pi}{2})/\overline{w} = \frac{\pi}{4} + \frac{\alpha - \gamma}{\overline{w}}$$

Iteration again yields

$$B(\frac{\pi}{4}) \approx \frac{\pi}{4} + \frac{\alpha - \gamma}{\overline{w}} - \frac{2\alpha(\alpha-\gamma)^2}{\overline{w}^3} + \frac{4\beta(\alpha-\gamma)}{\overline{w}^2}$$

Investigation shows that the two time iterations have been sufficiently accurate up to e^8. Expanding the equation above into a power series of the eccentricity, we get

$$B(\frac{\pi}{4}) = \frac{\pi}{4} + \frac{3}{8}e^2 + \frac{3}{16}e^4 + \frac{61}{768}e^6 + \frac{13}{512}e^8 + \cdots \tag{25}$$

Inserting eq. (25) into eq. (4) and omitting the tedious derivation procedure, we get the corresponding derivative value we expected

$$
\begin{aligned}
B'(\frac{\pi}{4}) &= \overline{w}(1 - e^2 \sin^2 B)^{3/2} = \overline{w}(1 - \frac{e^2}{2} - \frac{e^2}{2}\cos 2B)^{3/2} \\
&= 1 - \frac{21}{64}e^4 - \frac{21}{64}e^6 + \frac{3599}{16384}e^8 + \cdots
\end{aligned}
\tag{26}
$$

Secondly, a direct simple calculation to eq. (4) yields

$$B'(0) = \frac{dB}{dx}\Big|_{x=0} = \overline{w} \tag{27}$$

$$B'(\frac{\pi}{2}) = \frac{dB}{dx}\Big|_{x=\frac{\pi}{2}} = \overline{w}(1 - e^2)^{3/2} \tag{28}$$

Eqs. (25–28) have four interpolation constraints, being suitable to determine four undetermined coefficients. Inserting these constraints into eq. (23), we obtain one set of linear equation system as

$$
\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -4 & 0 & 8 \\ 2 & 4 & 6 & 8 \\ -2 & 4 & -6 & 8 \end{pmatrix}
\begin{pmatrix} a_2 \\ a_4 \\ a_6 \\ a_8 \end{pmatrix}
=
\begin{pmatrix} B(\frac{\pi}{4}) - \frac{\pi}{4} \\ B'(\frac{\pi}{4}) - 1 \\ B'(0) - 1 \\ B'(\frac{\pi}{2}) - 1 \end{pmatrix}
\tag{29}
$$

Therefore

$$
\begin{pmatrix} a_2 \\ a_4 \\ a_6 \\ a_8 \end{pmatrix}
=
\begin{pmatrix}
\frac{3}{4}[B(\pi/4) - \pi/4] + \frac{\overline{w}}{16}[1 - (1-e^2)^{3/2}] \\
-\frac{1}{8}[B'(\pi/4) - 1] + \frac{1}{16}[\overline{w} + \overline{w}(1-e^2)^{3/2} - 2] \\
-\frac{1}{4}[B(\pi/4) - \pi/4] + \frac{\overline{w}}{16}[1 - (1-e^2)^{3/2}] \\
\frac{1}{16}[B'(\pi/4) - 1] + \frac{1}{32}[\overline{w} + \overline{w}(1-e^2)^{3/2} - 2]
\end{pmatrix}
\tag{30}
$$

Expanding the relevant terms in eq. (30) into a series up to e^8, eq. (30) readily becomes

$$
\begin{pmatrix} a_2 \\ a_4 \\ a_6 \\ a_8 \end{pmatrix}
=
\begin{pmatrix}
\frac{3}{8}e^2 + \frac{3}{16}e^4 + \frac{213}{2048}e^6 + \frac{255}{4096}e^8 \\
\frac{21}{256}e^4 + \frac{21}{256}e^6 + \frac{35}{512}e^8 \\
\frac{151}{6144}e^6 + \frac{151}{4096}e^8 \\
\frac{881}{131072}e^8
\end{pmatrix}
\tag{31}
$$

For convenience with a comparison to eq. (22), a slight modification to eq. (31) results in

$$
\begin{pmatrix} a_2 \\ a_4 \\ a_6 \\ a_8 \end{pmatrix}
=
\begin{pmatrix}
\frac{3}{8}e^2 + \frac{3}{16}e^4 + \frac{106.5}{1024}e^6 + \frac{127.5}{2048}e^8 \\
\frac{21}{256}e^4 + \frac{21}{256}e^6 + \frac{560}{8192}e^8 \\
\frac{75.5}{3072}e^6 + \frac{75.5}{2048}e^8 \\
\frac{881}{131072}e^8
\end{pmatrix}
\tag{32}
$$

The coefficients of eq. (32) are either the same as or very near to those of eq. (22) except the coefficient of e^8. This indicates from the other hand that eq. (22) derived from Taylor series is accurate and correct though this kind of comparison seems not to be very logical. Two equations are derived from completely different approaches. Interpolation methods have no errors at data points and error distributes relatively uniformly while error in a Taylor series gradually increase with an increase of x. Therefore, the bigger difference on the coefficient of e^8 in eq. (22) and eq. (32) is also understandable and acceptable.

4 Numerical Examples and Their Error Analysis

Choose a reference ellipsoid as a Krasovsky ellipsoid with $a = 6378245\ m$, $\alpha = {}^{1}/_{298.3}$. In terms of the direct solution eq. (24), we first compute the length of the meridian arc at those specific points of $B = {}^{\pi}/_{8}, {}^{\pi}/_{4}, {}^{3\pi}/_{8}$. And then we re-compute the corresponding latitude value at points of $x(\pi/8)$, $x(\pi/4)$ and $x(3\pi/8)$ through eq. (22) and eq. (23) respectively.

Computation shows that errors for the inverse solution in terms of a Taylor expansion in section 2 are

$$\Delta B(\pi/8) = \pi/8 - B[x(\pi/8)] = -0.''68 \times 10^{-4}$$

$$\Delta B(\pi/4) = \pi/4 - B[x(\pi/4)] = -0.''48 \times 10^{-8}$$

$$\Delta B(3\pi/8) = 3\pi/8 - B[x(3\pi/8)] = -0.''68 \times 10^{-10}$$

at three points respectively.

Errors for the inverse solution in terms of the interpolation principle are

$$\Delta B(\pi/8) = -0.''23 \times 10^{-11}$$

$$\Delta B(\pi/4) = 0$$

$$\Delta B(3\pi/8) = -0.''11 \times 10^{-10}$$

From this example, it is clear that the inverse solution for the length of the meridian arc in terms of the interpolation principle is much more accurate than that in terms of a Taylor expansion. This is because the inverse solution in terms of interpolations has stronger control on the interval $[0, \pi/2]$. Its constraints are located not only at the initial point $x = 0$ but also at the middle point $x =$

$\pi/4$ and the end point $x = \pi/2$. The inverse solution derived in terms of a Taylor series has derivative constraints only at the initial point.

5 Conclusions

1. In terms of the derivative rule of implicit functions, one inverse solution for the length of the meridian arc is given in the paper. The expression of the general terms derived in the section, we think, is to some extent new and therefore makes a significant improvement on the theory of ellipsoid geodesy.
2. In terms of a Hermite interpolation, another inverse solution for the length of meridian arc is presented in the paper. As our knowledge, few authors implemented interpolation methods symbolically as we do here. This kind of inverse solution derived from a symbolic interpolation also enhances the basic theory of ellipsoid geodesy.
3. Both inverse solutions are sufficiently accurate up to 0."0001. However, the inverse solution in terms of a Hermite interpolation has much higher accuracy than that in terms of a Taylor expansion. Therefore, the former should be recommended in practice.

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Abstract

Two new solutions for an inverse problem on the length of the meridian arc are described in the paper. One is derived from a Taylor series expansion directly. Another is formulated by means of a Hermite interpolation. Both of the two solutions are sufficiently accurate up to 0."0001. Investigation in the paper show that the solution by means of a Hermite interpolation with much high accuracy should be recommended and implemented in practice.

Keywords: Ellipsoidal Geodesy, Projection, Interpolation, Taylor Expansion