Closed form solution of the overdetermined nonlinear 7 parameter datum transformation

Joseph L. Awange, Erik W. Grafarend

Der Gauss-Jacobi Algorithmus zur 7-Parameter-Transformation wird untersucht.

1 Introduction

Transformation of coordinates is a computational procedure that maps one set of coordinates in a given system into another coordinate system. This is achieved by translating the given system so as to cater for its origin with respect to the final system and rotating about its own axes so as to orient it to the final system. In addition to translation and rotation, scaling is performed in order to match the corresponding baseline length in the two systems. The *three translation parameters, three rotation parameters* and the *scale element* comprise the *7-parameter datum transformation elements* that are necessary to transform a set of three-dimensional coordinates from one system into another system.

The need for 7-parameter datum transformation has been on the rise following the increase in positioning using GPS and GLONASS satellites both in Geodesy and Photogrammetry. The bottleneck to the problem has been to obtain closed form solution to the problem. In practice, the users have been forced to rely on iterative procedures and least squares solution. Literature on the topic include: Harvey (1986), Wolfrum (1992), Abusali et al. (1994), Welsch *(1993)*, Abd-Elmotaal and El-Tokhey (1995), Grafarend et al. (1995), Birardi (1996), Kampmann (1996), Vanicek and Steeves (1996), Beranek (1997), Featherstone (1997), Grafarend and Syffus (1997, 1998), Okeke (1998), Grafarend and Okeke (1998), Awange (1999), Bazlov et al. (1999), Yang (1999), Baki Iz and Chen (1999), Grafarend and Awange (2000), Nitschke and Knickmeyer (2000), Grafarend et al. (2000) and Barsi (2001).

In Awange and Grafarend (2001), it was illustrated how the algebraic technique of *Groebner basis* solves explicitly the *nonlinear 7-parameters datum transformation equations* once they have been *converted* into algebraic (polynomial) form. In particular, the algebraic tool of *Groebner basis* provides symbolic solutions to the problem of *7-parameters datum transformation* and shows the *scale parameter* to be a *univariate polynomial* of order *four* while the rotation parameters are *linear functions* in scale and coordinates differences. Similar to the Gauss elimination technique in linear systems of equations, the *Groebner basis* approach eliminates several variables in a multivariate system of nonlinear equations in such a manner that the end product normally consist of *univariate polynomial* equations whose roots can be determined by existing programs such as the *roots* command in MATLAB. The present contribution considers the use of *Gauss-Jacobi combinatorial algorithm* in solving in a closed form the overdetermined *7-parameter datum transformation problem.* We employ the *Gröbner basis* in solving the minimum combinatorial subsets. We have used the *skew symmetric* matrix to construct the orthogonal matrix $\boldsymbol{X}_3$ in Section 3. Other approaches have been presented in Shut (1958/59) and Thompson (1959 a, b).

We organize the present study as follows; in Section 2 we present the *Gauss-Jacobi combinatorial algorithm* while Section 3 considers its application to the solution of the *overdetermined 7-parameter datum transformation problem*. In Section 4, we consider a case study.

2 Gauss-Jacobi combinatorial algorithm

In this Section we present the *Gauss-Jacobi combinatorial algorithm* which is neither iterative nor requires linearization of the observation equations for the solution of *nonlinear Gauss-Markov model.* Linearization is permitted only for the *nonlinear error propagation/variance-covariance propagation* in order to generate the dispersion matrix. We start by stating the *Gauss-Jacobi combinatorial lemma* in *Table* (2-1) and give its proofs in *Table* (2-2) and (2-3). *Theorem* (2-1) allows the applicability of the *Gauss-Jacobi combinatorial algorithm* in solving *nonlinear* Geodetic observation equations once they have been converted into *algebraic (polynomial) equations*.

Let us suppose that a new point P is to be positioned on a plane using distance measurements to n known stations. The *nonlinear* observation equations can be set in the form

$$S_i^2 = (x_i - x)^2 + (y_i - y)^2 \qquad 2\text{-}1$$

for $i = 1, \ldots, n$ and S_i being the Euclidean distances whose squares are considered as the observable instead

of the square root (e.g. GRAFAREND and SCHAFFRIN 1989, 1991). The linearized observation equations using Taylor series expansion to first order terms is given by

$$S_i^2 = S_{0i}^2 + \frac{\partial S_{0i}^2}{\partial x}\Delta x + \frac{\partial S_{0i}^2}{\partial y}\Delta y, \qquad 2\text{-}2$$

which can now be re-written taking into consideration the observational error term as

$$S_i^2 - S_{0i}^2 = \frac{\partial S_{0i}^2}{\partial x}\Delta x + \frac{\partial S_{0i}^2}{\partial y}\Delta y + \varepsilon_i. \qquad 2\text{-}3$$

If from (2-1) and (2-3) we write $a_i := \frac{\partial S_{0i}^2}{\partial x} = 2(x_i - x),\ b_i := \frac{\partial S_{0i}^2}{\partial y} = 2(y_i - y),\ y_i = S_i^2 - S_{0i}^2$, C. F. GAUSS and JACOBI (1841) combinatorial *lemma* is then stated as follows:

Table 2-1 (*Gauss-Jacobi combinatorial lemma*):

For n observations, let n equations of the form (2-3) be converted into algebraic form as

$$\begin{aligned} a_1x + b_1y - y_1 &= 0 \\ a_2x + b_2y - y_2 &= 0 \\ a_3x + b_3y - y_3 &= 0 \\ &\ldots. \end{aligned} \qquad 2\text{-}4$$

for the purpose of determining the coordinates x and y of the unknown point P, $y_i|i \in \{1, 2, \ldots n\}$ being the observable and $a_i, b_i|i \in \{1, 2, \ldots, n\}$ being the elements of the design matrix $\boldsymbol{A} \in R.^{n\times m}$ There will exist no set of solution $\{x, y\}$ from any combinatorial pair in the equations above that satisfy the entire system of equations. This is because the solution obtained from each *combinatorial pair* of equations differs from the others due to the unavoidable random measuring errors. If the solutions from the pair of the combinatorial equations are designated $x_{1,2}, x_{2,3}, \ldots$ and $y_{1,2}, y_{2,3}, \ldots$ with the subscript indicating the combinatorials, then the combined solution is the *sum of the weighted arithmetic* mean

$$x = \frac{p_{1,2}x_{1,2} + p_{2,3}x_{2,3} + \ldots.}{p_{1,2} + p_{2,3} + \ldots.}, \quad y = \frac{p_{1,2}y_{1,2} + p_{2,3}y_{2,3} + \ldots.}{p_{1,2} + p_{2,3} + \ldots.} \qquad 2\text{-}5$$

with $p_{1,2}, p_{2,3}, \ldots$ being the weights of the combinatorial solutions given by the square of the determinants as

$$\begin{aligned} p_{1,2} &= (a_1b_2 - a_2b_1)^2 \\ p_{2,3} &= (a_2b_1 - a_1b_2)^2 \end{aligned} \qquad 2\text{-}6$$

The results (2-5) coincides with the center of the error figure (*see Figure in Appendix* A) formed by the coordinates of the combinatorial solutions and are identical to those of *linear Gauss-Markov model.*

In order to prove the *Gauss-Jacobi combinatorial lemma*, we will consider two cases; one with the number of observations $n = 3$ and unknowns $m = 2$ and the other with $n = 4$ and $m = 2$. We first give the proof that the combinatorial solution coincides with the center of the error figure before proving that the results of the combinatorial approach are identical to those of *special linear Gauss-Markov model.*

Table 2-2 (*Proof 1: The Gauss-Jacobi combinatorial solution coincides with the center of the error figure*):

$n = 3,\ m = 2$:
We start by obtaining the equation of the center of the triangle formed by the three combinations $\{1, 2\}, \{2, 3\}, \{1, 3\}$ from (2-4) i. e.

$$\binom{3}{2} = \frac{3!}{2!(3-2)!} = 3,$$

leading to the solutions of the combinatorial pairs as

$$\begin{bmatrix} x \\ y \end{bmatrix}_{1,2} = (a_1b_2 - a_2b_1)^{-1}\begin{bmatrix} b_2 & -b_1 \\ a_2 & a_1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} =: d_1\begin{bmatrix} b_2y_1 - b_1y_2 \\ a_1y_2 - a_2y_1 \end{bmatrix} \qquad 2\text{-}7$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_{2,3} = (a_2b_3 - a_3b_2)^{-1}\begin{bmatrix} b_3 & -b_2 \\ a_3 & a_2 \end{bmatrix}\begin{bmatrix} y_2 \\ y_3 \end{bmatrix} =: d_2\begin{bmatrix} b_3y_2 - b_2y_3 \\ a_2y_3 - a_3y_2 \end{bmatrix} \qquad 2\text{-}8$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_{1,3} = (a_1b_3 - a_3b_1)^{-1}\begin{bmatrix} b_3 & -b_1 \\ a_3 & a_1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_3 \end{bmatrix} =: d_3\begin{bmatrix} b_3y_1 - b_1y_3 \\ a_1y_3 - a_3y_1 \end{bmatrix} \qquad 2\text{-}9$$

with d_1, d_2, d_3 in (2-7), (2-8), and (2-9) being the inverses of determinants of the design matrices $\boldsymbol{A}_{1,2}$, $\boldsymbol{A}_{2,3}$, and $\boldsymbol{A}_{1,3}$ respectively. The solutions above form a triangle whose center (centroid) is given by the summation of solutions (2-7), (2-8), and (2-9) divided by three as

$$centroid = \frac{d_1}{3}\begin{bmatrix} b_2y_1 - b_1y_2 \\ a_1Y_2 - a_2y_1 \end{bmatrix} + \frac{d_2}{3}\begin{bmatrix} b_3y_2 - b_2y_3 \\ a_2y_3 - a_3y_2 \end{bmatrix} + \frac{d_3}{3}\begin{bmatrix} b_3y_1 - b_1y_3 \\ a_1y_3 - a_3y_1 \end{bmatrix}. \tag{2-10}$$

Next, we show that the solution obtained from (2-5) coincides with (2-10) in which case the solution lies in the center of the triangle. From (2-5) and (2-6) we have the weighted combinatorial solutions of pairs $\{1, 2\}$, $\{2, 3\}$ and $\{1, 3\}$ as

$$\begin{bmatrix} x \\ y \end{bmatrix}_{1,2} = c_1\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \; c_1 := \frac{\sqrt{p_{1,2}}}{p_{1,2} + p_{2,3} + p_{1,3}} \tag{2-11}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_{2,3} = c_2\begin{bmatrix} b_3 & -b_2 \\ -a_3 & a_2 \end{bmatrix}\begin{bmatrix} y_2 \\ y_3 \end{bmatrix}, \; c_2 := \frac{\sqrt{p_{2,3}}}{p_{1,2} + p_{2,3} + p_{1,3}} \tag{2-12}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_{1,3} = c_3\begin{bmatrix} b_3 & -b_1 \\ -a_3 & a_1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_3 \end{bmatrix}, \; c_3 := \frac{\sqrt{p_{1,3}}}{p_{1,2} + p_{2,3} + p_{1,3}} \tag{2-13}$$

where c_1, c_2, c_3 in (2-11), (2-12), and (2-13) are the weights of the respective combinatorial pairs multiplied by the inverses of their respective determinants and $p_{1,2}$, $p_{2,3}$, $p_{1,3}$ are as given in (2-6). Expressing c_1, c_2, c_3 in terms of the coefficients of the centroid (2-10) we get

$$c_1 = \frac{p_{1,2}d_1}{p}, \; c_2 = \frac{p_{2,3}d_2}{p}, \; c_3 = \frac{p_{1,3}d_3}{p} \mid p = p_{1,2} + p_{2,3} + p_{1,3}. \tag{2-14}$$

If we now consider the weights to be unit, we have $p_{1,2} = p_{2,3} = p_{1,3} = 1$ and $p = 3$ thus leading to

$$c_1 = \frac{d_1}{3}, \; c_2 = \frac{d_2}{3}, \; c_3 = \frac{d_3}{3} \tag{2-15}$$

which are the coefficients of the centroid (2-10) confirming that the *Gauss-Jacobi combinatorial* solution (2-5) coincides with the center of the error triangle (*Appendix* A)

Table 2-2 (*Proof for $n = 4, m = 2$):*

$n = 4, m = 2$:
We start by obtaining the equation of the center of the six sided error figure formed by the six combinations $\{1, 2\}$, $\{2, 3\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 4\}$, $\{3, 4\}$ from (2-4) i.e.

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = 6,$$

leading to the solutions of the combinatorial pairs in addition (2-7), (2-8), and (2-9) as

$$\begin{bmatrix} x \\ y \end{bmatrix}_{1,4} = (a_1b_4 - a_4b_1)^{-1}\begin{bmatrix} b_4 & -b_1 \\ a_4 & a_1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_4 \end{bmatrix} =: d_4\begin{bmatrix} b_4y_1 - b_1y_4 \\ a_1y_4 - a_4y_1 \end{bmatrix} \tag{2-16}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_{2,4} = (a_2b_4 - a_4b_2)^{-1}\begin{bmatrix} b_4 & -b_2 \\ a_4 & a_2 \end{bmatrix}\begin{bmatrix} y_2 \\ y_4 \end{bmatrix} =: d_5\begin{bmatrix} b_4y_2 - b_2y_4 \\ a_2y_4 - a_4y_2 \end{bmatrix} \tag{2-17}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_{3,4} = (a_3b_4 - a_4b_3)^{-1}\begin{bmatrix} b_4 & -b_3 \\ a_4 & a_3 \end{bmatrix}\begin{bmatrix} y_3 \\ y_4 \end{bmatrix} =: d_6\begin{bmatrix} b_4y_3 - b_3y_4 \\ a_3y_4 - a_4y_3 \end{bmatrix} \tag{2-18}$$

with d_4, d_5, d_6 in (2-16), (2-17), and (2-18) being the inverses of the determinants of the design matrices $\boldsymbol{A}_{1,4}$, $\boldsymbol{A}_{2,4}$, and $\boldsymbol{A}_{3,4}$ respectively. The solutions above form a six sided figure whose center (centroid) is given by the summation of solutions (2-7), (2-8), (2-9), (2-16), (2-17), and (2-18) divided by six as

$$Centroid = \frac{d_1}{6}\begin{bmatrix} b_2y_1 - b_1y_2 \\ a_1y_2 - a_2y_1 \end{bmatrix} + \frac{d_2}{6}\begin{bmatrix} b_3y_2 - b_2y_3 \\ a_2y_3 - a_3y_2 \end{bmatrix} + \frac{d_3}{6}\begin{bmatrix} b_3y_1 - b_1y_3 \\ a_1y_3 - a_3y_1 \end{bmatrix} + \frac{d_4}{6}\begin{bmatrix} b_4y_1 - b_1y_4 \\ a_1y_4 - a_4y_1 \end{bmatrix} + \frac{d_5}{6}\begin{bmatrix} b_4y_2 - b_2y_4 \\ a_2y_4 - a_4y_2 \end{bmatrix} +$$

$$+ \frac{d_6}{6}\begin{bmatrix} b_4y_3 - b_3y_4 \\ a_3y_4 - a_4y_3 \end{bmatrix}. \tag{2-19}$$

We next show that the solution obtained from (2-5) coincides with (2-19) in which case the solution coincides with the center of the six sided figure. From (2-5) we have the weighted combinatorial solutions of equations $\{1, 4\}$, $\{2, 4\}$ and $\{3, 4\}$ as

$$\begin{bmatrix} x \\ y \end{bmatrix}_{1,4} = c_4 \begin{bmatrix} b_4 & -b_1 \\ -a_4 & a_1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_4 \end{bmatrix}, \; c_4 := \frac{\sqrt{p_{1,4}}}{p_{1,2} + p_{2,3} + p_{1,3} + p_{1,4} + p_{2,4} + p_{3,4}} \tag{2-20}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_{2,4} = c_5 \begin{bmatrix} b_4 & -b_2 \\ -a_4 & a_2 \end{bmatrix} \begin{bmatrix} y_2 \\ y_4 \end{bmatrix}, \; c_5 := \frac{\sqrt{p_{2,4}}}{p_{1,2} + p_{2,3} + p_{1,3} + p_{1,4} + p_{2,4} + p_{3,4}} \tag{2-21}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_{3,4} = c_6 \begin{bmatrix} b_4 & -b_3 \\ -a_4 & a_3 \end{bmatrix} \begin{bmatrix} y_3 \\ y_4 \end{bmatrix}, \; c_6 := \frac{\sqrt{p_{3,4}}}{p_{1,2} + p_{2,3} + p_{1,3} + p_{1,4} + p_{2,4} + p_{3,4}} \tag{2-22}$$

with c_4, c_5, c_6 in (2-20), (2-21), and (2-22) being the weights of the respective combinatorial pairs multiplied by the inverses of their respective determinants and $p_{1,4}, p_{2,4}, p_{3,4}$ as given in (2-6). Expressing $c_1, c_2, c_3, c_4, c_5, c_6$ in terms of the coefficients of (2-19) we get

$$c_1 = \frac{p_{1,2}d_1}{p}, \; c_2 = \frac{p_{2,3}d_2}{p}, \; c_3 = \frac{p_{1,3}d_3}{p}, \; c_4 = \frac{p_{1,4}d_4}{p}, \; c_5 = \frac{p_{2,4}d_5}{p}, \; c_6 = \frac{p_{3,4}d_6}{p} \mid p = p_{1,2} + \ldots + p_{3,4}. \tag{2-23}$$

If we now assume the weight to be unit, we have $p_{1,2} = \ldots = p_{3,4} = 1$ and $p = 6$ thus leading to

$$c_1 = \frac{d_1}{6}, \; c_2 = \frac{d_2}{6}, \; c_3 = \frac{d_3}{6}, \; c_4 = \frac{d_4}{6}, \; c_5 = \frac{d_5}{6}, \; c_6 = \frac{d_6}{6} \tag{2-24}$$

which are the coefficients of the centroid (2-19) confirming that the *Gauss-Jacobi combinatorial solutions* (2-5) coincides with the center of the six sided figure.

Table 2-3 (*Proof 2: For linear case, the results of the Gauss-Jacobi combinatorial solution and the linear Gauss-Markov model are identical*):

$$n = 3, \; m = 2: \; \boldsymbol{y} \in I\!R^3: \; \binom{n}{m} = \binom{3}{2} = 3.$$

We decompose the observation vector $\boldsymbol{y} = [y_1 \quad y_2 \quad y_3]'$ of observations as

$$\boldsymbol{y}_I = \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix}, \; \boldsymbol{y}_{II} = \begin{bmatrix} 0 \\ y_2 \\ y_3 \end{bmatrix}, \; \boldsymbol{y}_{III} = \begin{bmatrix} y_1 \\ 0 \\ y_3 \end{bmatrix}, \tag{2-25}$$

satisfying

$$\boldsymbol{y} = \frac{1}{2}\boldsymbol{y}_I + \frac{1}{2}\boldsymbol{y}_{II} + \frac{1}{2}\boldsymbol{y}_{III} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}. \tag{2-26}$$

$$n = 4, \; m = 2: \boldsymbol{y} \in I\!R^4: \; \binom{n}{m} = \binom{4}{2} = 6.$$

We decompose the observation vector $\boldsymbol{y} = [y_1 \quad y_2 \quad y_3 \quad y_4]'$ of observations as

$$\boldsymbol{y}_I = \begin{bmatrix} y_1 \\ y_2 \\ 0 \\ 0 \end{bmatrix}, \; \boldsymbol{y}_{II} = \begin{bmatrix} 0 \\ y_2 \\ y_3 \\ 0 \end{bmatrix}, \; \boldsymbol{y}_{III} = \begin{bmatrix} 0 \\ 0 \\ y_3 \\ y_4 \end{bmatrix}, \; \boldsymbol{y}_{IV} = \begin{bmatrix} y_1 \\ 0 \\ 0 \\ y_4 \end{bmatrix}, \; \boldsymbol{y}_V = \begin{bmatrix} 0 \\ y_2 \\ 0 \\ y_4 \end{bmatrix} \; \boldsymbol{y}_{VI} = \begin{bmatrix} y_1 \\ 0 \\ y_3 \\ 0 \end{bmatrix} \tag{2-27}$$

satisfying

$$\boldsymbol{y} = \frac{1}{3}\boldsymbol{y}_I + \frac{1}{3}\boldsymbol{y}_{II} + \frac{1}{3}\boldsymbol{y}_{III} + \frac{1}{3}\boldsymbol{y}_{IV} + \frac{1}{3}\boldsymbol{y}_V + \frac{1}{3}\boldsymbol{y}_{VI} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}. \tag{2-28}$$

With (2-42) we have

$$\hat{\boldsymbol{\xi}} = \frac{1}{3}(\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}_I + \boldsymbol{y}_{II} + \boldsymbol{y}_{III} + \boldsymbol{y}_{IV} + \boldsymbol{y}_V + \boldsymbol{y}_{VI}), \tag{2-29}$$

$$\hat{\xi} = (\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{\Sigma}^{-1} \sum_{i=1}^{\binom{m}{n}} \alpha_i \boldsymbol{y}_i \tag{2-30}$$

which is written as

$$\hat{\xi} = (\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Y}\alpha \tag{2 – 31}$$

and finally as

$$\hat{\xi} = \alpha(\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Y}\mathbf{1}, \tag{2 – 32}$$

where

$$\alpha = \frac{1}{n-m+1} = \frac{1}{3}, \boldsymbol{Y} = [\boldsymbol{y}_I \quad \boldsymbol{y}_{II} \quad \boldsymbol{y}_{III} \quad \boldsymbol{y}_{IV} \quad \boldsymbol{y}_V \quad \boldsymbol{y}_{VI}], \mathbf{1} = [111111]'$$

Having stated and given proof of the *Gauss-Jacobi combinatorial lemma* for linear equations, we state below the *theorem* that allows the solution of *nonlinear equations* in Geodesy.

Table 2-4} *(Theorem* 2-1)

Given algebraic (polynomial) observational equations (n observations, where n is the dimension of the observation space $\mathbf{Y}$ of order l in m variables (unknowns) (m is the dimension of the parameter space $\mathbf{X}$, the application of least squares solution (LESS) to the algebraic observation equations gives) $(2l-1)$ (as the order of the set of *nonlinear algebraic normal equations*. There exists m normal equations of the polynomial order) $(2l-1)$ to be solved.
Prof: Given nonlinear algebraic equations $f_i \in k\{\xi_1, \ldots, \xi_m$ expressed as

$$\begin{bmatrix} f_1 \in k\{\xi_1, \ldots, \xi_m\} \\ f_1 \in k\{\xi_1, \ldots, \xi_m\} \\ \vdots \\ f_n \in k\{\xi_1, \ldots, \xi_m\} \ . \end{bmatrix} \tag{2 – 33}$$

and the order considered as l, we write the objective function to be minimized as

$$\| \boldsymbol{f} \|^2 = f_1^2 + \ \ldots \ + f_n^2 | \forall f_i \in k\{\xi_1, \ldots, \xi_m\} \tag{2 – 34}$$

and obtain the partial derivatives (first derivatives) of (2-34) with respect to the unknown variables $\{\xi_1, \ldots, \xi_m\}$. The order of (2-34) which is l^2 then reduces to $(2l-1)$ upon differentiating the objective function with respect to the variables $\xi_1, \ldots, \xi_m$. Thus resulting in m normal equations of the polynomial order $(2l-1)$.

Example (pseudo-ranging):

For pseudo-ranging or distance equations, the order of the polynomials in the algebraic observational equations is $l = 2$. If we take the "pseudo-ranges squared" or "distances squared", a necessary procedure in order to make the observational equations "algebraic" or "polynomial", and implement LESS, the objective function which is of order $l = 4$ reduces by one to order $l = 3$ upon differentiating once. The normal equations are of order $l = 3$ as expected.

The significance of the theorem above is that by using the *Gauss-Jacobi combinatorial approach* to solve the *nonlinear Gauss-Markov model*, all observation equations of geodetic interest are successfully converted to "*algebraic*" or "*polynomial*" equations.

With the above definitions, lemma and theorem, we proceed to execute the *Gauss-Jacobi combinatorial lemma* in two steps as follows:

- Step 1: Combinatorial minimal subsets of observations are constructed and rigorously solved by means of the *Gröbner bases* (Awange and Grafarend 2001).
- Step 2: The combinatorial solution points of step 1 are reduced to their final adjusted values by means of their weighted means generated via the *nonlinear error propagation law* also known in this case as the *nonlinear variance-covariance propagation*.

Construction of the minimal combinatorial subsets:

Since $n > m$, we construct the minimal combinatorial sets solvable in closed form using *Gröbner basis* (Awange and Grafarend 2001). We begin by the following elementary definitions:

Definition 2-1 (Permutation):

Given a set S with elements $\{i, j, k\} \in S$, the arrangement obtained by placing $\{i, j, k\} \in S$ in some sequence is called *permutation*. If we choose any of the elements say i first, then each of the remaining elements j, k can be put in the

second position, while the third position is occupied by the unused letter either j or k. For the set S, the following permutations can be made:

$$\begin{bmatrix} ijk & ikj & jik \\ jki & kij & kji \end{bmatrix} \tag{2-35}$$

From (2-35) there exist three ways of filling the first position, two ways of filling the second position and one way of filling the third position. Thus the number of *permutations* is given by $3 \times 2 \times 1 = 6$. In general, for n different elements, the number of permutation is equal to $n \times \ldots \times 3 \times 2 \times 1 = n!$

Definition 2-2 (Combination):

If for n elements only m elements are used for the permutation, then we have a *combination* of the mth order. If we follow the definition above, then the first position can be filled in n ways, the second in $n-1$ ways and the mth in $n-(m-1)$ ways. In (2-35) above the combinations are identical and contain the same elements in different sequences. If the arrangement is to be neglected, then we have for n elements, a combination of mth order being given by

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{n(n-1)\ \ldots\ (n-m+1)}{m \times\ \ldots \times 3 \times 2 \times 1}. \tag{2-36}$$

Given n nonlinear equations to be solved, we first form $\binom{n}{m}$ minimal combinatorial subsets each consisting of m elements (where m is the number of the unknown elements). Each minimal combinatorial subset is solved using the algebraic procedures of *Gröbner bases*. In geodesy, the number of elements n normally consist of the observations in the vector $\boldsymbol{y}$, while the number of elements m normally consist of the unknown fixed parameters in the vector $\boldsymbol{\xi}$.

Adjustment of the combinatorial minimal subsets solutions:

Once the *combinatorial minimal subsets* have been solved using the *Gröbner bases* approach, the resulting sets of solution are considered as pseudo-observations. For each combinatorial, the obtained minimal subset solutions considered as pseudo-observations are used as the approximate values to generate the dispersion matrix via the *nonlinear error propagation law/variance-covariance propagation* (*e.g.* Grafarend and Schaffrin, 1993, pp.469–471) as follows:

From the nonlinear geodetic observation equations that have been converted into its algebraic (polynomial) form, the *combinatorialminimalsubsets* will consist of polynomials $f_1, \ldots, f_m \in k[x_1, \ldots, x_m]$ with $\{x_1, \ldots, x_m\}$ being the unknown variables (fixed parameters) to be determined and $\{y_1, \ldots, y_n\}$ the known variables comprising the observations or pseudo-observations. We write the polynomials as

$$\left.\begin{aligned} f_1 &:= g(x_1, \ldots, x_m, y_1, \ldots, y_n) = 0 \\ f_2 &:= g(x_1, \ldots, x_m, y_1, \ldots, y_n) = 0 \\ &\vdots \\ f_m &:= g(x_1, \ldots, x_m, y_1, \ldots, y_n) = 0 \end{aligned}\right\} \tag{2-37}$$

which is expressed in matrix form as

$$f := g(x,\ y) = 0 \tag{2-38}$$

where the unknown variables $\{x_1, \ldots, x_m\}$ are placed in a vector x and the known variables $\{y_1, \ldots, y_n\}$ are placed in the vector y. Next, we implement the error propagation from the observations (pseudo-observations) $\{y_1, \ldots, y_n\}$ to the parameters $\{x_1, \ldots, x_m\}$ that are to be explicitly determined namely characterized by the *first moments*, the expectation $E\{\boldsymbol{x}\} = \mu_x$ and $E\{\boldsymbol{y}\} = \mu_y$, as well as the *second moments*, the variance-covariance matrices/dispersion matrices $D\{\boldsymbol{x}\} = \boldsymbol{\Sigma}_x$ and $D\{\boldsymbol{y}\} = \boldsymbol{\Sigma}_y$. From Grafarend and Schaffrin (1993, pp. 470–471), we have upto nonlinear terms

$$D\{\boldsymbol{x}\} = \boldsymbol{J}_x^{-1}\boldsymbol{J}_y\boldsymbol{\Sigma}_y\boldsymbol{J}_y'(\boldsymbol{J}_x^{-1})' \tag{2-39}$$

with $\boldsymbol{J}_x,\ \boldsymbol{J}_y$ being the partial derivatives of (2-38) with respect to $\boldsymbol{x},\ \boldsymbol{y}$ respectively at the Taylor points $(\mu_x,\ \mu_y)$. The unknown values $\{x_1, \ldots, x_m\} \in \boldsymbol{x}$ appearing in the Jacobi matrices $\boldsymbol{J}_x,\ \boldsymbol{J}_y$ are obtained from *Gröbner basis* solution of the *nonlinear system of equations* (2-37).

The sub-matrices variance-covariance matrix for the individual combinatorials $\boldsymbol{\Sigma}_1,\ \boldsymbol{\Sigma}_2,\ \boldsymbol{\Sigma}_3, \ldots, \boldsymbol{\Sigma}_k$ (where k is the number of combinations) obtained via (2-39) are combined using the *sparse matrix techniques* to form the variance-covariance/dispersion matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_1 & & & & & \\ & \boldsymbol{\Sigma}_2 & & & & \\ & & \boldsymbol{\Sigma}_3 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \boldsymbol{\Sigma}_k \end{bmatrix} \tag{2-40}$$

of the entire k combinations. In (2-40) above, we assume the respective *minimal combinatorial sets* to be uncorrelated with one another (i.e. the first combinatorial set leading to $\boldsymbol{\Sigma}_1$ to be uncorrelated with the second combinatorial set leading to $\boldsymbol{\Sigma}_2$. Within a particular combinatorial set (e.g. the first combinatorial set leading to $\boldsymbol{\Sigma}_1$), correlation between the elements is taken into consideration. This is because the unknown parameters $\{x_1, \ldots, x_m\} \in \boldsymbol{x}$ appearing in the Jacobi matrices $\boldsymbol{J}_x$, $\boldsymbol{J}_y$ are obtained from *Gröbner bases* solution of the nonlinear algebraic equations (2-37) in a closed form. We define the *special linear Gauss-Markov model* as follows:

Definition 2-3 (*Special linear Gauss-Markov model*):

Given a real $n \times 1$ random vector $\boldsymbol{y} \in \mathbb{R}^n$ of observations, a real $m \times 1$ vector $\xi \in \mathbb{R}^m$ of unknown fixed parameters over a real $n \times m$ coefficient matrix $\boldsymbol{A} \in \mathbb{R}^{n\times m}$, a real $n \times n$ positive-definite dispersion matrix $\boldsymbol{\Sigma}$, the functional model

$$\boldsymbol{A}\xi = E\{\boldsymbol{y}\},\ E\{\boldsymbol{y}\} \in R(\boldsymbol{A}),\ rk\boldsymbol{A} = m,\ \boldsymbol{\Sigma} = D\{\boldsymbol{y}\},\ rk\boldsymbol{\Sigma} = n \tag{2-41}$$

is called the a *special linear Gauss-Markov* model with full rank.

The unknown vector ξ of fixed parameters in the *special linear Gauss-Markov model* is normally estimated by Best Linear Uniformly Unbiased Estimator (BLUUE) as

$$\hat{\boldsymbol{\xi}} = (\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{y} \tag{2-42}$$

with its regular dispersion matrix

$$D\{\hat{\boldsymbol{\xi}}\} = (\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A})^{-1}. \tag{2-43}$$

The dispersion matrix (variance-covariance matrix) $\boldsymbol{\Sigma}$ is *unknown* and is obtained by means of estimators of type MINQUE, BIQUUE or BIQE as in Rao (1967, 1971, 1973 and 1978), Rao and Kleffe (1979), Schaffrin (1983), and Grafarend (1985). From (2-40), the obtained dispersion matrix $\boldsymbol{\Sigma}$ is then used in the *linear Gauss-Markov model* (2-42) to obtain the estimates $\hat{\boldsymbol{\xi}}$of the unknown parameters ξ with the minimum combinatorial solution considered as pseudo-observations in the vector $\boldsymbol{y}$ of observations while the design matrix $\boldsymbol{A}$ comprises of integer values 1 being the coefficients of the unknowns. The dispersion matrix $D\{\hat{\boldsymbol{\xi}}\}$ of the estimates $\hat{\boldsymbol{\xi}}$is obtained via (2-43).
In the event that $\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A}$ is not regular (i.e. $\boldsymbol{A}$ has a rank deficiency), the rank deficiency can be overcome by procedures such as those presented by Mittermayer (1972), Grafarend and Schaffrin (1974), Grafarend and Schaffrin (1993, pp. 107–165), Brunner (1979), Peremulter (1979), Meissl (1982), Grafarend and Sanso (1985) and Koch (1999, pp. 181–197) among others. This *algorithm* described above is here referred to as the *Gauss-Jacobi combinatorial algorithm*. In *Appendix B,* we present for completeness purpose the error propagation based on the nonlinear random effect model (multivariate) where we illustrate the effect of the biased term.

3 Solution of the overdetermined 7-parameters datum transformation problem

In Awange and Grafarend (2001), the univariate polynomial for solving the scale parameters and the linear functions in terms of scale and coordinate differences for solving the rotation parameters were derived using *Gröbner basis*. These are presented in *Tables* (3-1) and (3-2) as:

Table 3-1 (*Univariate polynomial for computing scale parameter*):

$$a_4x_1^4 + a_3x_1^3 + a_2x_1^2 + a_1x_1 + a_0 = 0$$

$$a_4 = (X_{13}Y_{12}^2Y_{23} + X_{12}^2X_{13}Y_{23} - X_{12}^2X_{23}Y_{13} - X_{12}Y_{13}Z_{12}Z_{23} - X_{23}Y_{12}^2Y_{13} + X_{13}Y_{12}Z_{12}Z_{23} - X_{23}Y_{12}Z_{12}Z_{13} + X_{12}Y_{23}Z_{12}Z_{13})$$

$$a_3 = (c_{12}X_{13}Y_{12}Z_{23} - b_{13}X_{12}Z_{12}Z_{23} - c_{13}X_{12}Y_{23}Z_{12} + b_{23}X_{13}Y_{12}^2 - a_{23}X_{12}^2Y_{13} + c_{23}X_{12}Y_{13}Z_{12} + c_{12}X_{12}Y_{23}Z_{13} + a_{13}Y_{12}^2Y_{23} - b_{12}X_{23}Z_{12}Z_{13} - a_{23}Y_{12}^2Y_{13} + b_{12}X_{13}Z_{12}Z_{23} - c_{12}X_{12}Y_{13}Z_{23} + a_{13}X_{12}^2Y_{23} - c_{23}X_{13}Y_{12}Z_{12} - a_{23}Y_{12}Z_{12}Z_{13} + c_{13}X_{23}Y_{12}Z_{12} - b_{13}X_{23}Y_{12}^2 - b_{13}X_{12}^2X_{23} + b_{23}X_{12}Z_{12}Z_{13} + a_{13}Y_{12}Z_{12}Z_{23} + b_{23}X_{12}^2X_{13} + a_{12}Y_{23}Z_{12}Z_{13} - a_{12}Y_{13}Z_{12}Z_{23} - c_{12}X_{23}Y_{12}Z_{13})$$

$$a_2 = (a_{13}b_{23}X_{12}^2 + b_{12}^2X_{23}Y_{13} + b_{12}c_{13}X_{23}Z_{12} - c_{12}c_{23}X_{13}Y_{12} + b_{13}c_{23}X_{12}Z_{12} - a_{23}b_{12}Z_{12}Z_{13} + a_{12}^2X_{23}Y_{13} - b_{12}^2X_{13}Y_{23} - a_{12}^2X_{13}Y_{23} - a_{23}b_{13}X_{12}^2 + a_{13}b_{23}Y_{12}^2a_{23}b_{13}Y_{12}^2 + a_{12}b_{23}Z_{12}Z_{13} + a_{23}c_{13}Y_{12}Z_{12} + a_{12}c_{12}Y_{23}Z_{13} - b_{12}c_{12}X_{23}Z_{13} - b_{23}c_{13}X_{12}Z_{12} - a_{12}b_{13}Z_{12}Z_{23} - a_{23}c_{12}Y_{12}Z_{13} + a_{12}c_{23}Y_{13}Z_{12} + b_{12}c_{12}X_{13}Z_{23} - a_{12}c_{12}Y_{13}Z_{23} - a_{13}c_{23}Y_{12}Z_{12} - c_{12}c_{13}X_{12}Y_{23} + c_{12}c_{13}X_{23}Y_{12} + c_{12}c_{23}X_{12}Y_{13} - b_{13}c_{12}X_{12}Z_{23} + b_{23}c_{12}X_{12}Z_{13} + a_{13}b_{12}Z_{12}Z_{23} - a_{12}c_{13}Y_{23}Z_{12} + a_{13}c_{12}Y_{12}Z_{23} - b_{12}c_{23}X_{13}Z_{12})$$

$$a_1 = (-a_{12}b_{13}c_{12}Z_{23} + b_{12}c_{12}c_{13}X_{23} + b_{12}^2b_{13}X_{23} - a_{13}b_{12}^2Y_{23} - a_{12}^2b_{23}X_{13} + a_{23}c_{12}c_{13}Y_{12} - a_{13}c_{12}c_{23}Y_{12} + a_{12}b_{13}c_{23}Z_{12} - a_{12}^2a_{13}Y_{23} - b_{23}c_{12}c_{13}X_{12} + a_{12}b_{23}c_{12}Z_{13} - a_{23}b_{12}c_{12}Z_{13} - b_{12}^2b_{23}X_{13} + a_{23}b_{12}c_{13}Z_{12} - a_{12}c_{12}c_{13}Y_{23} + a_{12}^2b_{13}X_{23} + a_{12}^2a_{23}Y_{13} - a_{12}b_{23}c_{13}Z_{12} - a_{13}b_{12}c_{23}Z_{12} - b_{12}c_{12}c_{23}X_{13} + b_{13}c_{12}c_{23}X_{12} + a_{12}c_{12}c_{23}Y_{13} + a_{23}b_{12}^2Y_{13} + a_{13}b_{12}c_{12}Z_{23})$$

$$a_0 = a_{12}b_{13}c_{12}c_{23} - a_{13}b_{12}^2b_{23} + a_{12}^2a_{23}b_{13} - a_{12}^2a_{13}b_{23} - a_{12}b_{23}c_{12}c_{13} + a_{23}b_{12}^2b_{13} + a_{23}b_{12}c_{12}c_{13} - a_{13}b_{12}c_{12}c_{23}$$

Once the admissible value of scale parameter has been chosen amongst the four roots above as $x_1 \in \mathbb{R},^+$ the elements of the *skew symmetric* matrix $\boldsymbol{S}$ can then be obtained via the linear functions in *Table* (3-2):

Table 3-2 (*Linear functions for computing the parameters of the skew-symmetric matrix S*):

$$f(a) = (-x_1^3X_{13}Y_{12}Z_{12} - b_{12}x_1^2X_{13}Z_{12} - a_{13}b_{12}c_{12} - a_{13}x_1^2Y_{12}Z_{12} + b_{13}x_1^2X_{12}Z_{12} + c_{12}x_1^2X_{12}Y_{13} - a_{13}c_{12}x_1Y_{12} + a_{12}b_{13}c_{12} + a_{12}b_{13}x_1Z_{12} - a_{13}b_{12}x_1Z_{12} - b_{12}c_{12}x_1X_{13} + a_{12}x_1^2Y_{13}Z_{12} + x_1^3X_{12}Y_{13}Z_{12} + a_{12}c_{12}x_1Y_{13} - c_{12}x_1^2X_{13}Y_{12} + b_{13}c_{12}x_1X_{12})a + (-a_{12}^2a_{13} - a_{12}c_{13}x_1Z_{12} + a_{12}x_1^2Z_{12}Z_{13} + a_{12}c_{12}x_1Z_{13} - c_{12}c_{13}x_1X_{12} - a_{12}c_{12}c_{13} + x_1^3X_{13}Y_{12}^2 - c_{13}x_1^2X_{12}Z_{12} + x_1^3X_{12}^2X_{13} + x_1^3X_{12}Z_{12}Z_{13} - a_{12}^2x_1X_{13} - b_{12}^2x_1X_{13} - a_{13}b_{12}^2 + a_{13}x_1^2X_{12}^2 + a_{13}x_1^2Y_{12}^2 + c_{12}x_1^2X_{12}Z_{13})$$

$$f(b) = (-b_{12}c_{12}x_1X_{13} + b_{13}c_{12}x_1X_{12} - a_{13}b_{12}x_1Z_{12} - x_1^3X_{13}Y_{12}Z_{12} - a_{13}c_{12}x_1Y_{12} + a_{12}b_{13}c_{12} + a_{12}x_1^2Y_{13}Z_{12} - a_{13}x_1^2Y_{12}Z_{12} + a_{12}b_{13}x_1Z_{12} + x_1^3X_{12}Y_{13}Z_{12} - a_{13}b_{12}c_{12} + c_{12}x_1^2X_{12}Y_{13} + a_{12}c_{12}x_1Y_{13} - b_{12}x_1^2X_{13}Z_{12} - c_{12}x_1^2X_{13}Y_{12} + b_{13}x_1^2X_{12}Z_{12})b + (-a_{12}^2b_{13} - c_{12}c_{13}x_1Y_{12} + b_{13}x_1^2Y_{12}^2 + c_{12}x_1^2Y_{12}Z_{13} + x_1^3Y_{12}Z_{12}Z_{13} + x_1^3X_{12}^2Y_{13} - b_{12}c_{13}x_1Z_{12} + b_{12}x_1^2Z_{12}Z_{13} - b_{12}^2b_{13} - a_{12}^2x_1Y_{13} + b_{13}x_1^2X_{12}^2 - c_{13}x_1^2Y_{12}Z_{12} + b_{12}c_{12}x_1Z_{13} + x_1^3Y_{12}^2Y_{13} - b_{12}^2x_1Y_{13} - b_{12}c_{12}c_{13})$$

$$f(c) = (-b_{12}x_1X_{13} - a_{13}x_1Y_{12} - x_1^2X_{13}Y_{12}a_{13}b_{12} + x_1^2X_{12}Y_{13} + a_{12}b_{13} + a_{12}x_1Y_{13} + b_{13}x_1X_{12})c + (-a_{12}a_{13} - b_{12}b_{13} + x_1^2Z_{12}Z_{13} + x_1^2Y_{12}Y_{13} - c_{12}c_{13} + a_{13}x_1X_{12} + b_{13}x_1Y_{12} - a_{12}x_1X_{13} + x_1^2X_{12}X_{13} + c_{12}x_1Z_{13} - b_{12}x_1Y_{13} - c_{13}x_1Z_{12})\ .$$

From the elements of the *skew symmetric* matrix $\boldsymbol{S}$ in *Table* (3-2), the rotation matrix $\boldsymbol{X}_3$ from which the Cardan rotation angles are deduced is formed. The translation elements $\boldsymbol{x}_2$ can then be computed by substituting the scale parameter x_1 and the rotation matrix $\boldsymbol{X}_3$ in the 7-parameter datum transformation equations. Three sets of translation parameters are then obtained for the three points under consideration and the mean taken.

From three points in both system involved in the transformation process, 9 nonlinear equations

$$\left[\begin{array}{l} f_1 := x_1X_1 - x_1cY_1 + x_1bZ_1 + X_0 - a_1 - cb_1 + bc_1 = 0 \\ f_2 := x_1cX_1 + x_1Y_1 - x_1aZ_1 + Y_0 + ca_1 - b_1 - ac_1 = 0 \\ f_3 := -x_1bX_1 + x_1aY_1 + x_1Z_1 + Z_0 - ba_1 + ab_1 - c_1 = 0 \\ f_4 := x_1X_2 - x_1cY_2 + x_1bZ_2 + X_0 - a_2 - cb_2 + bc_2 = 0 \\ f_5 := x_1cX_2 + x_1Y_2 - x_1aZ_2 + Y_0 + ca_2 - b_2 - ac_2 = 0 \\ f_6 := -x_1bX_2 + x_1aY_2 + x_1Z_2 + Z_0 - ba_2 + ab_2 - c_2 = 0 \\ f_7 := x_1X_3 - x_1cY_3 + x_1bZ_3 + X_0 - a_3 - cb_3 + bc_3 = 0 \\ f_8 := x_1cX_3 + x_1Y_3 - x_1aZ_3 + Y_0 + ca_3 - b_3 - ac_3 = 0 \\ f_9 := -x_1bX_3 + x_1aY_3 + x_1Z_3 + Z_0 - ba_3 + ab_3 - c_3 = 0 \end{array}\right. \tag{3-1}$$

are formed with $\{x_i, y_i, z_i\} := \{a_i, b_i, c_i\} | i \in \{1, 2, 3\}$, equations $\{f_1, f_2, f_3\}$ being equations formed from the *first point* with coordinates in both systems, $\{f_4, f_5, f_6\}$ being equations formed from the *second point* with coordinates in both systems and $\{f_7, f_8, f_9\}$ being the equations formed from the *third point* with coordinates in both systems. Applying (2-36) in (3-1) we have 36 combinatorials each with 7 equations that can be solved in closed form using *Groebner basis* approach to give symbolic solutions in the form given by *Tables* (3-1) and (3-2). For combinatorial $\{f_1, f_2, f_3, f_4, f_5, f_6, f_9\}$ for example, we eliminate the translation parameters by differencing:

$$\begin{bmatrix} f_{14} := f_1 - f_4 = x_1X_{12} - x_1cY_{12} + x_1bZ_{12} - a_{12}c - b_{12} + bc_{12} \\ f_{25} := f_2 - f_5 = x_1cX_{12} + x_1Y_{12} - x_1aZ_{12} + ca_{12} - b_{12} - ac_{12} \\ f_{39} := f_3 - f_9 = -x_1bX_{13} + x_1aY_{13} + x_1Z_{13} - ba_{13} + ab_{13} - c_{13} \\ f_{69} := f_6 - f_9 = -x_1bX_{23} + x_1aY_{23} + x_1Z_{23} - ba_{23} + ab_{23} - c_{23} \end{bmatrix} \tag{3-2}$$

with

$$\left.\begin{matrix} X_{ij} = X_i - X_j & Y_{ij} = Y_i - Y_j & Z_{ij} = Z_i - Z_j \\ a_{ij} = a_i - a_j & b_{ij} = b_i - b_j & c_{ij} = c_i - c_j \end{matrix}\right\} | i,j \in \{1,2,3\}, i \neq j.$$

On applying *Groebner basis* on (3-2) we obtain the results of *Tables* (3-1) and (3-2). This is repeated for all the 36 combinatorials. The dispersion matrix for each combinatorial is obtained via the *nonlinear error propagation law/ variance-covariance propagation law*. From the algebraic system of equations (3-1), we have the Jacobi matrices as

$$\boldsymbol{J}_x = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial a} & \frac{\partial f_1}{\partial b} & \frac{\partial f_1}{\partial c} & \frac{\partial f_1}{\partial X_0} & \frac{\partial f_1}{\partial Y_0} & \frac{\partial f_1}{\partial Z_0} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial a} & \frac{\partial f_2}{\partial b} & \frac{\partial f_2}{\partial c} & \frac{\partial f_2}{\partial X_0} & \frac{\partial f_2}{\partial Y_0} & \frac{\partial f_2}{\partial Z_0} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial a} & \frac{\partial f_3}{\partial b} & \frac{\partial f_3}{\partial c} & \frac{\partial f_3}{\partial X_0} & \frac{\partial f_3}{\partial Y_0} & \frac{\partial f_3}{\partial Z_0} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial a} & \frac{\partial f_4}{\partial b} & \frac{\partial f_4}{\partial c} & \frac{\partial f_4}{\partial X_0} & \frac{\partial f_4}{\partial Y_0} & \frac{\partial f_4}{\partial Z_0} \\ \frac{\partial f_5}{\partial x_1} & \frac{\partial f_5}{\partial a} & \frac{\partial f_5}{\partial b} & \frac{\partial f_5}{\partial c} & \frac{\partial f_5}{\partial X_0} & \frac{\partial f_5}{\partial Y_0} & \frac{\partial f_5}{\partial Z_0} \\ \frac{\partial f_6}{\partial x_1} & \frac{\partial f_6}{\partial a} & \frac{\partial f_6}{\partial b} & \frac{\partial f_6}{\partial c} & \frac{\partial f_6}{\partial X_0} & \frac{\partial f_6}{\partial Y_0} & \frac{\partial f_6}{\partial Z_0} \\ \frac{\partial f_9}{\partial x_1} & \frac{\partial f_9}{\partial a} & \frac{\partial f_9}{\partial b} & \frac{\partial f_9}{\partial c} & \frac{\partial f_9}{\partial X_0} & \frac{\partial f_9}{\partial Y_0} & \frac{\partial f_9}{\partial Z_0} \end{bmatrix} \tag{3-3}$$

and

$$\boldsymbol{J}_y = \begin{bmatrix} \frac{\partial f_1}{\partial a_1} & \frac{\partial f_1}{\partial b_1} & \frac{\partial f_1}{\partial c_1} & \frac{\partial f_1}{\partial a_2} & \frac{\partial f_1}{\partial b_2} & \frac{\partial f_1}{\partial c_2} & \frac{\partial f_1}{\partial a_3} & \frac{\partial f_1}{\partial b_3} & \frac{\partial f_1}{\partial c_3} & \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial Z_1} & \cdot & \cdot & \cdot & \frac{\partial f_1}{\partial X_3} & \frac{\partial f_1}{\partial Y_3} & \frac{\partial f_1}{\partial Z_3} \\ \frac{\partial f_2}{\partial a_1} & \frac{\partial f_2}{\partial b_1} & \frac{\partial f_2}{\partial c_1} & \frac{\partial f_2}{\partial a_2} & \frac{\partial f_2}{\partial b_2} & \frac{\partial f_2}{\partial c_2} & \frac{\partial f_2}{\partial a_3} & \frac{\partial f_2}{\partial b_3} & \frac{\partial f_2}{\partial c_3} & \frac{\partial f_2}{\partial X_1} & \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial Z_1} & \cdot & \cdot & \cdot & \frac{\partial f_2}{\partial X_3} & \frac{\partial f_2}{\partial Y_3} & \frac{\partial f_2}{\partial Z_3} \\ \frac{\partial f_3}{\partial a_1} & \frac{\partial f_3}{\partial b_1} & \frac{\partial f_3}{\partial c_1} & \frac{\partial f_3}{\partial a_2} & \frac{\partial f_3}{\partial b_2} & \frac{\partial f_3}{\partial c_2} & \frac{\partial f_3}{\partial a_3} & \frac{\partial f_3}{\partial b_3} & \frac{\partial f_3}{\partial c_3} & \frac{\partial f_3}{\partial X_1} & \frac{\partial f_3}{\partial y_1} & \frac{\partial f_3}{\partial Z_1} & \cdot & \cdot & \cdot & \frac{\partial f_3}{\partial X_3} & \frac{\partial f_3}{\partial Y_3} & \frac{\partial f_3}{\partial Z_3} \\ \frac{\partial f_4}{\partial a_1} & \frac{\partial f_4}{\partial b_1} & \frac{\partial f_4}{\partial c_1} & \frac{\partial f_4}{\partial a_2} & \frac{\partial f_4}{\partial b_2} & \frac{\partial f_4}{\partial c_2} & \frac{\partial f_4}{\partial a_3} & \frac{\partial f_4}{\partial b_3} & \frac{\partial f_4}{\partial c_3} & \frac{\partial f_4}{\partial X_1} & \frac{\partial f_4}{\partial y_1} & \frac{\partial f_4}{\partial Z_1} & \cdot & \cdot & \cdot & \frac{\partial f_4}{\partial X_3} & \frac{\partial f_4}{\partial Y_3} & \frac{\partial f_4}{\partial Z_3} \\ \frac{\partial f_5}{\partial a_1} & \frac{\partial f_5}{\partial b_1} & \frac{\partial f_5}{\partial c_1} & \frac{\partial f_5}{\partial a_2} & \frac{\partial f_5}{\partial b_2} & \frac{\partial f_5}{\partial c_2} & \frac{\partial f_5}{\partial a_3} & \frac{\partial f_5}{\partial b_3} & \frac{\partial f_5}{\partial c_3} & \frac{\partial f_5}{\partial X_1} & \frac{\partial f_5}{\partial y_1} & \frac{\partial f_5}{\partial Z_1} & \cdot & \cdot & \cdot & \frac{\partial f_5}{\partial X_3} & \frac{\partial f_5}{\partial Y_3} & \frac{\partial f_5}{\partial Z_3} \\ \frac{\partial f_6}{\partial a_1} & \frac{\partial f_6}{\partial b_1} & \frac{\partial f_6}{\partial c_1} & \frac{\partial f_6}{\partial a_2} & \frac{\partial f_6}{\partial b_2} & \frac{\partial f_6}{\partial c_2} & \frac{\partial f_6}{\partial a_3} & \frac{\partial f_6}{\partial b_3} & \frac{\partial f_6}{\partial c_3} & \frac{\partial f_6}{\partial X_1} & \frac{\partial f_6}{\partial y_1} & \frac{\partial f_6}{\partial Z_1} & \cdot & \cdot & \cdot & \frac{\partial f_6}{\partial X_3} & \frac{\partial f_6}{\partial Y_3} & \frac{\partial f_6}{\partial Z_3} \\ \frac{\partial f_9}{\partial a_1} & \frac{\partial f_9}{\partial b_1} & \frac{\partial f_9}{\partial c_1} & \frac{\partial f_9}{\partial a_2} & \frac{\partial f_9}{\partial b_2} & \frac{\partial f_9}{\partial c_2} & \frac{\partial f_9}{\partial a_3} & \frac{\partial f_9}{\partial b_3} & \frac{\partial f_9}{\partial c_3} & \frac{\partial f_9}{\partial X_1} & \frac{\partial f_9}{\partial y_1} & \frac{\partial f_9}{\partial Z_1} & \cdot & \cdot & \cdot & \frac{\partial f_9}{\partial X_3} & \frac{\partial f_9}{\partial Y_3} & \frac{\partial f_9}{\partial Z_3} \end{bmatrix} \tag{3-4}$$

where the doted points in $\boldsymbol{J}_y$ represents the partial derivative of (3-1) with respect $\{X_2, Y_2, Z_2\}$. From the dispersion $\boldsymbol{\Sigma}_y$ of the vector of observations $\boldsymbol{y}$ and with $\boldsymbol{J} = \boldsymbol{J}_x^{-1}\boldsymbol{J}_y$, the dispersion matrix $\boldsymbol{\Sigma}_x$ is then obtained from (2-39). For scale, elements of the symmetric matrix $\boldsymbol{S}$ and the translation parameters, 36 solutions are formed from the combinatorial solutions. The adjusted values and their dispersion matrix are then obtained from (2-42) and (2-43) respectively.

4 Case study

We consider Cartesian coordinates of three stations given in the Local and Global Reference Systems (WGS 84) as in *Tables* (4-1) and (4-2). Desired are the seven parameters of datum transformation. Taking the three coordinates in both systems and applying the *Groebner bases algorithm*, the 7-transformation parameters are obtained for each minimum combinatorial set. *Tables* (4-3) to (4-5) present the 36 combinatorial solutions for scale, elements of the symmetric matrix $\boldsymbol{S}$ and the translation parameters. *Table* (4-6) presents the final adjusted values of the 7-parameter datum transformation which are used to transform the Cartesian coordinates from the *Local Reference System (Table 4-1)* to the *Global Reference System* (WGS 84, *Table* 4-2) in *Table* (4-7). *Figures* (4-1) indicate the scatter of the computed 36 minimal combinatorial solutions of scale (indicated by doted points (●)) around for the adjusted value indicated by a line (–). *Figures* (4-2) indicate the scatter of the computed 36 minimal combinatorial solutions of translation and rotation parameters (indicated by doted points (●)) around the adjusted values indicated by a star (*). The Figures clearly identifies the outlying combinations from which the respective (suspected outlying points) points can be deduced. The computed residuals in *Table* (4-7) are in cm range.

Table 4-1: Coordinates for system A (Local system)

Station Name	$X(m)$	$Y(m)$	$Z(m)$
Solitude	4157222.543	664789.307	4774952.099
Buoch Zeil	4149043.336	688836.443	4778632.188
Hohenneuffen	4172803.511	690340.078	4758129.701

Table 4-2: Coordinates for system B (WGS 84)

Station Name	$X(m)$	$Y(m)$	$Z(m)$
Solitude	4157870.237	664818.678	4775416.524
Buoch Zeil	4149691.049	688865.785	4779096.588
Hohenneuffen	4173451.354	690369.375	4758594.075

Table 4-3: Computed scale parameters from the 36 combinatorials

Combinatorial No.	Scale parameter k
1	0.999998565715064913
2	0.999998565715064735
3	0.999998565715064735
4	0.999998240604468335
5	0.999996823422180902
6	0.999998470409936679
7	0.999995284163028763
8	1.00000356554232339
9	0.999998010466988063
10	0.999997998947098665
11	0.999993750297076466
12	0.999998408954924578
13	1.00000109346871512
14	1.00000109346871512
15	1.00000109346871512
16	0.999998240604468335
17	1.00000503985655431
18	0.999998486675850273
19	0.999998240604468335
20	1.00000336322609784
21	0.999996729172877607
22	1.00000188266490153
23	1.00000356554232339
24	1.00000022602868865
25	1.00000356554232339
26	1.00000427807473113
27	1.0000036004614663
28	1.00000445362887169
29	0.999997888891338604
30	1.00000424701318402
31	0.999998408954924578
32	0.999999176340667794
33	1.00000331522783558
34	0.999998408954924578
35	1.0000036004614663
36	1.0000036004614663

Adjusted value of scale Scale $(k) = 1.00000090492535$

Table 4-4: Computed parameters of the rotation matrix from the 36 combinatorials

Combinatorial No.	Rotation $\mathbf{X}_1(a)$	Rotation $\mathbf{X}_2(b)$	Rotation $\mathbf{X}_3(c)$
1	0.384525773466933	– 1.6278694989721	– 0.311468547818445
2	– 0.565353110685831	1.16480981918273	0.115912124082905
3	– 0.32575151119894	0.460373977029875	0.00810678889607737
4	– 0.0776568063125417	– 1.70637511413975	– 0.300675717914113
5	– 2.0923481993924	– 2.04858775571586	– 0.253630949369419
6	– 0.422385919377009	0.32313136627419	– 0.00620995458978237
7	0.757636441784519	– 2.42028232666349	– 0.202507870917581
8	– 0.183948687500674	– 0.420540756047311	– 0.477479404271789
9	– 0.341499375718228	0.558202850299325	0.0620329990499175
10	0.284851982160257	– 1.28222824444608	– 0.610326329342833
11	– 0.462316280121334	1.30880539508379	– 2.8506473028306
12	– 0.330197387608202	0.487993546894341	– 0.0849563040111603
13	– 4.35716681370504	– 6.71363605719077	3.35481460137189
14	– 0.883683381947344	– 1.01748084629324	– 0.39540129589626
15	– 0.2540601349819	0.0150115113108204	– 1.07518493280313
16	– 0.0776568063125418	– 1.70637511422695	– 0.300675717914113
17	9.58824153078028	– 0.0645458162514901	– 0.5264291800642
18	– 0.446786377134148	0.46678229487655	0.0146328203120828
19	– 0.0776568063125418	– 1.70637511422695	– 0.300675717926752
20	– 1.52496150857494	– 0.469393217686175	– 0.470763467006947
21	2.18964467179755	0.94641890259548	– 2.23738045790161
22	0.00739206415346873	– 0.826911429014064	– 0.421602753602655
23	– 0.183948687500672	– 0.420540756047312	– 0.477479402278048
24	– 0.278662143613748	0.167844472669476	– 0.153144192881302
25	– 0.183948687500674	– 0.420540756047311	– 0.477479404317212
26	– 0.163740074828404	– 0.546080917299573	– 0.305201383198236
27	0.0475058735368953	– 0.412108980427943	– 0.478638585922024
28	0.0674289683151228	– 0.206091880589807	– 0.506966213897953
29	0.449944998913048	– 1.75739787005437	– 0.751347159647055
30	– 0.164616193658147	– 0.54060969933337	3.38088824205619
31	– 0.33019738759767	0.487993547167012	– 0.0849563039791073
32	– 1.48134617567308	– 1.57063427666969	0.898339895911698
33	– 0.191049242283465	– 0.376437517550037	– 1.89471647353154
34	– 0.33019738759767	0.487993547163035	– 0.0849563039791073
35	– 8.11792247680056	– 0.928969600556758	6.56723183817428
36	– 0.182958128491611	– 0.426693200967067	– 0.279773319879728

Adjusted value of Rotation $\mathbf{X}_1(a)$ =-0.294966996619736″, $\mathbf{X}_2(b)$ =-0.527171590194436″, $\mathbf{X}_3(c)$ =-0.27335443653029″

Table 4-5: Computed parameters of the translation from the 36 combinatorials

Combinatorial No.	Translation $\Delta X(m)$	Translation $\Delta Y(m)$	Translation $\Delta Z(m)$
1	690.33742571886	45.4886838696742	437.143261938045
2	627.158992685067	14.9006923738634	496.599951577373
3	643.095278462395	22.61630993026	481.602276626353
4	693.541140403443	34.8029360371099	438.637680995899
5	707.506380900896	– 11.7770258785846	445.152263400145
6	646.618618558471	20.7349709426441	479.608397302839
7	722.674749559102	54.1565822228634	435.58571524142
8	641.067744006248	32.2821680382282	439.516634970903
9	643.320455654214	21.5428851056301	486.27603228949
10	683.713673337052	49.5964906264174	447.146710213895
11	634.060909938843	80.3902100157381	522.135625931745
12	642.801160233095	24.4970673647864	482.921791067347
13	809.56141631538	– 139.841051471497	338.188597926249
14	665.428211446851	16.1273583586638	441.582892823343
15	639.302881129241	44.3986705058099	460.325104619997
16	693.541140405616	34.8029360371099	438.637680994036
17	626.539742621283	258.281752226137	407.389150573251
18	643.297400787473	19.7392231184834	482.508373385295
19	693.541140405616	34.8029360373427	438.637680994036
20	643.061382897819	1.26896452267344	443.926241738722
21	632.076486780929	127.32833669843	491.857655530485
22	657.651283561407	36.727305914353	438.71775193233
23	641.067744012612	32.2821679979873	439.516634970903
24	642.421964127881	25.8260882681934	467.62672117694
25	641.067744006092	32.2821680391207	439.516634970903
26	641.576449853523	28.7897982891106	433.51895386291
27	640.723648210056	37.6349521726758	438.755571188095
28	632.316304285234	38.0857489665893	438.774367357604
29	694.695618908387	56.3337675860773	437.543908162974
30	653.755398005402	– 45.5459574035679	433.780457522099
31	642.801160226731	24.4970673644372	482.921791072935
32	690.469869507632	– 22.4800491917801	441.547943815589
33	636.407744477658	60.8696676800416	441.623619170549
34	642.801160227042	24.4970673644372	482.921791072624
35	675.953566662191	– 293.311530960685	455.312848093919
36	641.717834656127	28.2941832978589	439.222708388542

Adjusted value of translation $\Delta X = 655.2738$m $\Delta Y = 27.4065$m $\Delta Z = 450.4307$m

Table 4-6: Computed 7-parameter datum transformation using Gauss-Jacobi combinatorial algorithm

Transformation parameter	Value	Root-mean-square	unit
Scale $k-1$	0.90492535	0.498669714	[ppm]
Rotation $\mathbf{X}_1(a)$	– 0.29496700	0.068980382	["]
Rotation $\mathbf{X}_2(b)$	– 0.52717159	0.072002244	["]
Rotation $\mathbf{X}_3(c)$	– 0.27335444	0.067150234	["]
Translation ΔX	655.2738	4.0518	[m]
Translation ΔY	27.4065	4.997	[m]
Translation ΔZ	450.4307	3.6986	[m]

Table 4-8: Transformed Cartesian coordinates of System A (Table 4-1) into System B (Table 4-2) using the 7-datum transformation parameters of Table (4-6) computed by Gauss-Jacobi algorithm

Site	$X(m)$	$Y(m)$	$Z(m)$
System A: Solitude	4157222.5430	664789.3070	4774952.0990
System B	4157870.2370	664818.6780	4775416.5240
Transformed value	4157870.2560	664818.6341	4775416.5250
Residual	– 0.0190	0.0439	– 0.0010
System A: Buoch Zeil	4149043.3360	688836.4430	4778632.1880
System B	4149691.0490	688865.7850	4779096.5880
Transformed value	4149691.0640	688865.8079	4779096.5621
Residual	– 0.0150	– 0.0229	0.0259
System A: Hohenneuffen	4172803.5110	690340.0780	4758129.7010
System B	4173451.3540	690369.3750	4758594.0750
Transformed value	4173451.3149	690369.3835	4758594.1151
Residual	0.0391	– 0.0085	– 0.0400

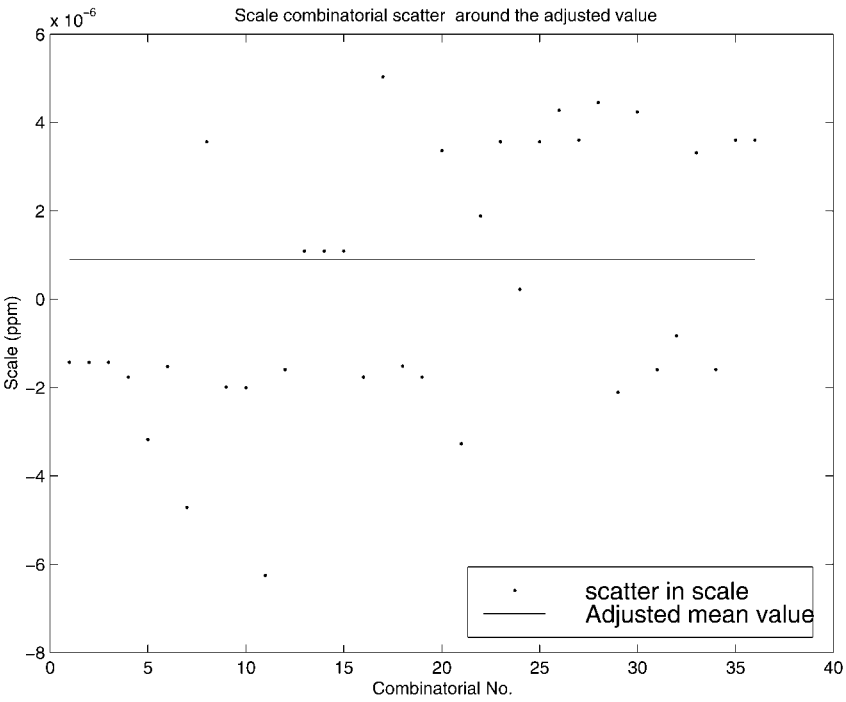
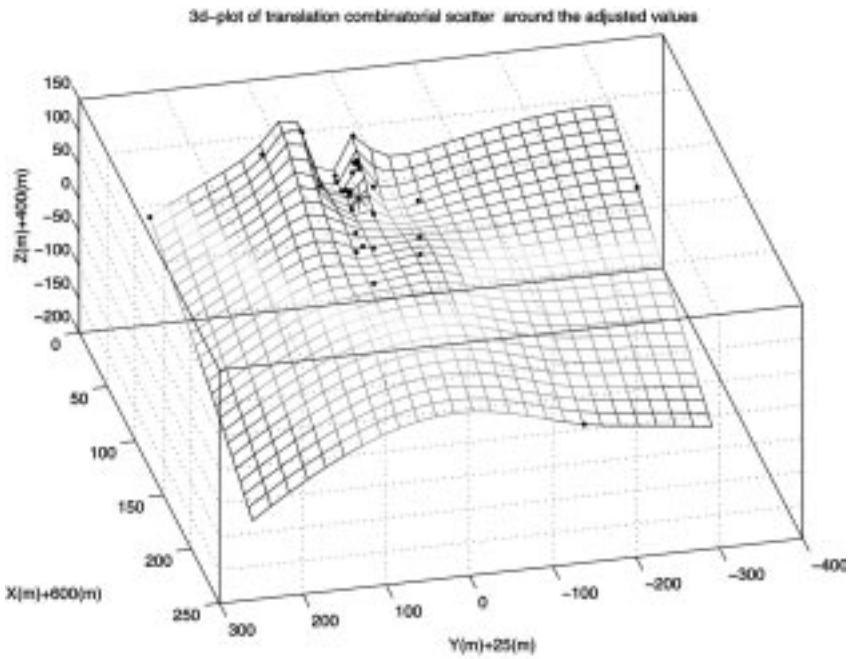


Fig. 4-1: Scatter of the 36 computed scale around the adjusted values



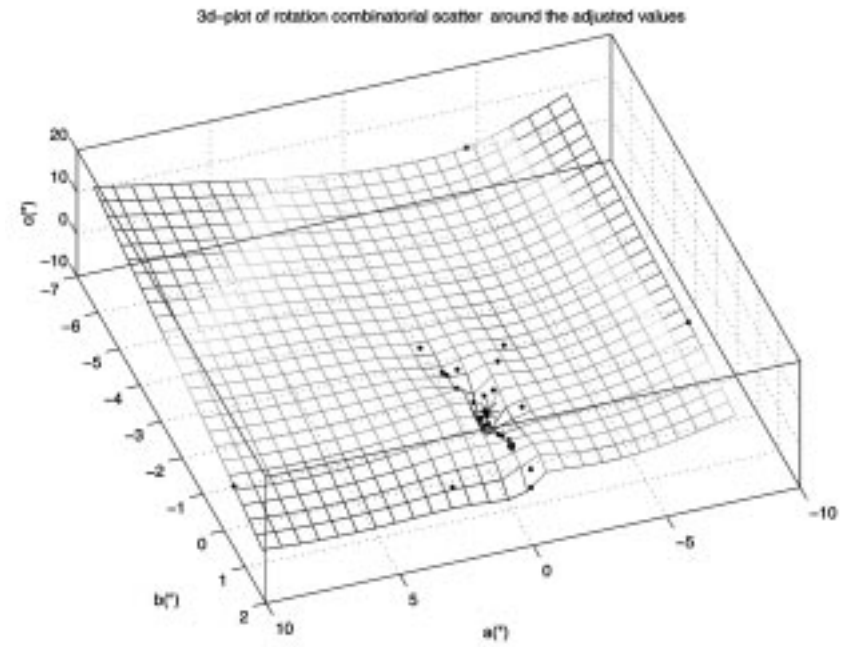


Fig. 4-2: Scatter of the 36 computed rotations around the adjusted values

Appendix A: C. F. Gauss Combinatorial approach

CARL FRIEDRICH GAUSS

WERKE

NEUNTER BAND.

G 6 - 4

Bücherei des Geodätischen Institu
der Techn. Hochschule Stuttgart
Nr. 3774

HERAUSGEGEBEN
VON DER
KÖNIGLICHEN GESELLSCHAFT DER WISSENSCHAFTEN
ZU
GÖTTINGEN.
IN COMMISSION BEI B. G. TEUBNER IN LEIPZIG.
1903.

BESTIMMUNG
DES
BREITENUNTERSCHIEDES
ZWISCHEN DEN
STERNWARTEN VON GÖTTINGEN UND ALTONA
DURCH
BEOBACHTUNGEN AM RAMSDENSCHEN ZENITHSECTOR
VON

CARL FRIEDRICH GAUSS,
RITTER DES GUELPHEN- UND DANNEBROG-ORDENS; K. GROSSBR. HANNOVERSCHEN HOFRATH;
PROFESSOR DER ASTRONOMIE UND DIRECTOR DER STERNWARTE IN GÖTTINGEN;
MITGLIED DER AKADEMIEN UND SOCIETÄTEN VON BERLIN, COPENHAGEN, EDINBURG, GÖTTINGEN,
LONDON, MÜNCHEN, NEAPEL, PARIS, PETERSBURG, STOCKHOLM,
DER AMERIKANISCHEN, ITALIENISCHEN, KURLÄNDISCHEN, LONDONER ASTRONOMISCHEN U. A.

GÖTTINGEN,
BEI VANDENHOECK UND RUPRECHT.
1828.

IX. 1

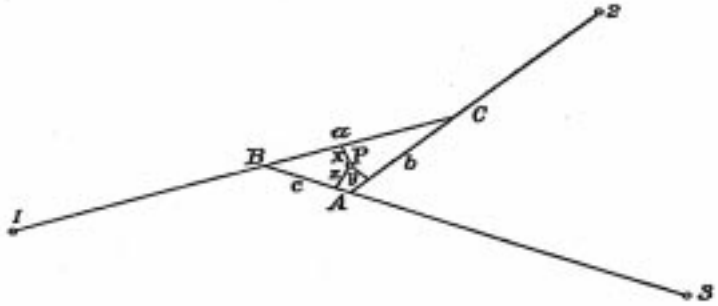
NACHLASS.

[1.]

Endresultat für den Ort eines Punktes in einer Ebene, der von drei bekannten aus angeschnitten ist.

Es bedeuten 10, 20, 30 die drei beobachteten Richtungen [nach P] und α, β, γ die entsprechenden Entfernungen.

Die drei einzelnen Resultate aus den Combinationen 2—3, 1—3, 1—2 seien A, B, C, zugleich die Winkel des durch jene gebildeten Dreiecks; die ihnen gegenüber stehenden Seiten a, b, c.



Perpendikel von dem gesuchten Orte auf a, b, c seien x, y, z. S doppelter Flächeninhalt des Dreiecks.

Es sind dann

$$\frac{x}{\alpha}, \quad \frac{y}{\beta}, \quad \frac{z}{\gamma}$$

die übrig bleibenden Fehler, also

$$\frac{xx}{\alpha\alpha}+\frac{yy}{\beta\beta}+\frac{zz}{\gamma\gamma} \quad \text{Minimum}$$

und

$$ax+by+cz=S.$$

Also werden x, y, z proportional den Grössen $\alpha\alpha a$, $\beta\beta b$, $\gamma\gamma c$;

$$x=\frac{\alpha\alpha aS}{\alpha\alpha aa+\beta\beta bb+\gamma\gamma cc},$$

etc.

[Bezeichnet (ABC) die Fläche des Dreiecks ABC, u. s. f., so ist

$$\begin{aligned} S = 2(ABC) &= (\alpha\alpha aa+\beta\beta bb+\gamma\gamma cc)k \\ 2(BPC) &= \alpha\alpha aak \\ 2(APC) &= \beta\beta bbk \\ 2(APB) &= \gamma\gamma cck, \end{aligned}$$

wo k die Correlate der Bedingungsgleichung ist. P ist der durch die Perpendikel x, y, z bestimmte Punkt.

Folglich wird, wenn A, B, C, P die complexen Grössen bedeuten, denen die Eckpunkte des Dreiecks ABC und der Punkt P entsprechen:

$$(\alpha\alpha aa+\beta\beta bb+\gamma\gamma cc)P=\alpha\alpha aaA+\beta\beta bbB+\gamma\gamma ccC.]$$

Es folgt hieraus, dass das Endresultat*)

$$\frac{\alpha\alpha aaA+\beta\beta bbB+\gamma\gamma ccC}{\alpha\alpha aa+\beta\beta bb+\gamma\gamma cc}$$

also ein Mittel aus den drei partiellen Resultaten A, B, C ist, indem man diesen die Gewichte

$$\alpha\alpha aa, \qquad \beta\beta bb, \qquad \gamma\gamma cc$$

beilegt, oder

$$\alpha\alpha \sin A^2, \qquad \beta\beta \sin B^2, \qquad \gamma\gamma \sin C^2.$$

Offenbar ist hier A zugleich der Winkel zwischen 20 und 30, u. s. f.

19

Appendix B: Error propagation (Nonlinear random effect model -univariate)

Consider a function $y = f(z)$ where y is a scalar valued observation and z a random effect. Three cases can be specified as follows:

Case 1
(μ_z assumed to be known):
By Taylor series expansion we have

$$f(z) = f(\mu_z) + \frac{1}{1!}f'(\mu_z)(z-\mu_z) + \frac{1}{2!}f''(\mu_z)(z-\mu_z)^2 + = O(3)$$

$$E\{y\} = E\{f(z)\} = f(\mu_z) + \frac{1}{2!}f''(\mu_z)E\{(z-\mu_z)^2\} + = O(3)$$

leading to (cf. E. Grafarend and B. Schaffrin 1983, p. 470)

$$\boxed{E\{y\} = f(\mu_z) + \frac{1}{2!}f''(\mu_z)\sigma_z^2 + O(3)}\ .$$

$$E\{(y-E\{y\})^2\} = E\{[f'(\mu_z)(z-\mu_z) + \frac{1}{2!}f''(\mu_z)(z-\mu_z)^2 + O(3) - \frac{1}{2!}f''(\mu_z)\sigma_z^2 - O(3)]^2\},$$

hence $E\{[y-E\{y\}][y-E\{y\}]\}$ is given by

$$\sigma_y^2 = f'^2(\mu_z)\sigma_z^2 - \frac{1}{4}f''^2(\mu_z)\sigma_z^4 + f'f''(\mu_z)E\{(z-\mu_z)^3\} + \frac{1}{4}f''^2E\{(z-\mu_z)^4\} + = O(3).$$

Finally if z is *quasi-normally distributed*, we have from E. Grafarend and B. Schaffrin (1983, p.468) that $\pi_3 = E\{(z-\mu_z)^3\} = 0$ and $\pi_4 = E\{(z-\mu_z)^4\} = 3\pi_2^2 = 3\sigma_z^4$ leading to

$$\boxed{\sigma_y^2 = f'^2(\mu_z)\sigma_z^2 + \frac{1}{2}f''^2(\mu_z)\sigma_z^4 + O(3)}$$

Case 2
(μ_z unknown, but ξ_0 known as a fixed effect approximation (this implies in E. Grafarend and B. Schaffrin 1983, p. 470 that $\xi_0 \neq \mu_z$)):
By Taylor series expansion we have

$$f(z) = f(\xi_0) + \frac{1}{1!}f'(\xi_0)(z-\xi_0) + \frac{1}{2!}f''(\xi_0)(z-\xi_0)^2 + O(3)$$

using

$$\xi_0 = \mu_z + (\xi_0 - \mu_z) \rightarrow z - \xi_0 = z - \mu_z + (\mu_z - \xi_0)$$

we have

$$f(z) = f(\xi_0) + \frac{1}{1!}f'(\xi_0)(z-\mu_z) + \frac{1}{1!}f'(\xi_0)(\mu_z-\xi_0) + \frac{1}{2!}f''(\xi_0)(z-\mu_z)^2 + \frac{1}{2!}f''(\xi_0)(\mu_z-\xi_0)^2 +$$
$$f''(\xi_0)(z-\mu_z)(\mu_z-\xi_0) + O(3)$$

and

$$E\{y\} = E\{f(z)\} = f(\xi_0) + f'(\xi_0)(\mu_z-\xi_0) + \frac{1}{2}f''(\xi_0)\sigma_z^2 + \frac{1}{2}f''(\xi_0)(\mu_z-\xi_0)^2 + O(3)$$

leading to $E\{[y-E\{y\}][y-E\{y\}]\}$ as

$$\sigma_y^2 = f'^2(\xi_0)\sigma_z^2 + f'f''(\xi_0)E\{(z-\mu_z)^3\} + 2f'f''(\xi_0)\sigma_z^2(\mu_z-\xi_0) + \frac{1}{4}f''^2(\xi_0)E\{(z-\mu_z)^4\} +$$

$$f''^2(\xi_0)E\{(z-\mu_z)^3\}(\mu_z-\xi_0) - \frac{1}{4}f''^2(\xi_0)\sigma_z^4 + f''^2(\xi_0)\sigma_z^2(\mu_z-\xi_0)^2 + O(3)$$

and with z being *quasi-normally distributed*, thus $\pi_3 = E\{(z-\mu_z)^3\} = 0$ and $\pi_4 = E\{(z-\mu_z)^4\} = 3\pi_2^2 = 3\sigma_z^4$, we have

$$\boxed{\sigma_y^2 = f'^2(\xi_0)\sigma_z^2 + 2f'f''(\xi_0)\sigma_z^2(\mu_z-\xi_0) + \frac{1}{2}f''^2(\xi_0)\sigma_z^4 + f''^2(\xi_0)\sigma_z^2(\mu_z-\xi_0)^2 + O(3)}$$

with the *first* and *third* terms (on the right hand side) being the right hand side terms of case 1 (cf. E. Grafarend and B. Schaffrin 1983, p. 470).

Case 3 (μ_z unknown, but z_0 known as a random effect approximation):
By Taylor series expansion we have

$$f(z) = f(\mu_z) + \frac{1}{1!}f'(\mu_z)(z-\mu_z) + \frac{1}{2!}f''(\mu_z)(z-\mu_z)^2 + \frac{1}{3!}f'''(\mu_z)(z-\mu_z)^3 + O(4)$$

changing

$$z - \mu_z = z_0 - \mu_z = z_0 - E\{z_0\} - (\mu_z - E\{z_0\})$$

and the *initial bias*

$$-(\mu_z - E\{z_0\}) = E\{z_0\} - \mu_z =: \beta_0$$

leads to

$$\boxed{z - \mu_z = z_0 - E\{z_0\} + \beta_0}\ .$$

If we also change $f(\mu_z) = f(z_0) + f'(z_0)(\mu_z - z_0) + O(2), f'(\mu_z) = f'(z_0) + f''(z_0)(\mu_z - z_0) + O(3)$ etc, the *derivatives became random effects* and cannot any longer be separated in $E\{f'(\mu_z)(z-\mu_z)\} = f'(\mu_z)E\{(z-\mu_z)\}$ etc. Consider

$$(z-\mu_2)^2 1^2 = (z_0 - E(\{z_0\})^2 + \beta_0^2 + 2(z_0 - E(\{z_0\})\beta_0$$

we have

$$f(z) = f(\mu_z) + \frac{1}{1!}f'(\mu_z)(z_0 - E\{z_0\}) + \frac{1}{1!}f'(\mu_z)\beta_0 + \frac{1}{2!}f''(\mu_z)(z_0 - E(\{z_0\})^2 + \frac{1}{2!}f''(\mu_z)\beta_0^2 +$$
$$f''(\mu_z)(z_0 - E\{z_0\})\beta_0 + O(3)$$

$$E\{y\} = f(\mu_z) + f'(\mu_z)\beta_0 + \frac{1}{2}f''(\mu_z)\sigma_{z_0}^2 + \frac{1}{2}f''(\mu_z)\beta_0^2 + O(3)$$

leading to $E\{[y - E\{y\}][y - E\{y\}]\}$as

$$\sigma_y^2 = f'^2(\mu_z)\sigma_{z_0}^2 + f'f''(\mu_z)E\{(z_0 - E\{z_0\})^3\} + 2f'f''(\mu_z)\sigma_{z_0}^2\beta_0 + \frac{1}{4}f''^2(\mu_z)E\{(z_0 - E\{z_0\}^4\} +$$
$$f''^2(\mu_z)E\{(z_0 - E\{z_0\})^3\}\beta_0 + f''^2(\mu_z)\sigma_{z_0}^2\beta_0^2 + O(3)$$

and with z_0 being *quasi-normally distributed*, thus $\pi_3 = E\{(z_0 - E\{z_0\})^3\} = 0$
and $\pi_4 = E\{(z_0 - E\{z_0\})^4\} = 3\pi_2^2 = 3\sigma_{z_0}^4$, we have

$$\boxed{\sigma_y^2 = f'^2(\mu_z)\sigma_{z_0}^2 + 2f'f''(\mu_z)\sigma_{z_0}^2\beta_0 + \frac{3}{4}f''^2(\mu_z)\sigma_{z_0}^4 + f''^2(\mu_{z_0})\sigma_z^2\beta_0^2 + O(3)}\ .$$

References

Abd-Elmotaal, H.; El-Tokhey, M.: (1995) Effect of spherical approximation on datum transformation. Manuscripta Geodaetica 20: 469-474

Abusali, P. A. M.; Schutz, B. E.; Tapley, B. D.: (1994) Transformation between SLR/VLBI and WGS-84 frames. Bulletin Geodesique 69: 61-72

Awange, L. J.: (1999) Partial procrustes solution of the three-dimensional orientation problem from GPS/LPS observations. In: F. Krumm and V. S. Schwarze. Quo vadis geodesia. . .? Festschrift to E. W. Grafarend on the occasion of his 60th birthday, Report Nr. 1999.6-1

Awange, L. J.; Grafarend, E.: (2001) Closed from solution of the nonlinear 7 parameter datum transformation by Groebner basis. Submitted to the Journal of Geodesy

Baki, I. z H.; Chen, Y. Q.; (1999) Tailored datum transformation model for locally distributed data. Journal of Surveying Engineering 125: 25-35

Barsi, A.; (2001) Performing coordinate transformation by artificial neural network. Allgemeine Vermessungs-Nachrichten 108: 134-137

Bazlov, Y. A.; Galazin, V. F.; Kaplan, B. L.; Maksimov, V. G.; Rogozin, V. P.; (1999) Propagating PZ 90 and WGS 94 transformation parameters. GPS Solutions 3(1): 13-16

Beranek, M.; (1997) Überlegungen zur Normalverteilung und theoretische Analyse der 3- und 4-Parameter-Transformation. Allgemeine Vermessungs-Nachrichten 104: 137-141

Birardi, G.; (1996) The future global geodetic network and the transformation of local onto WGS coordinates. Bullettino di Geodesia e Scienze Affini 55: 49-56

Brunner, F. K.; (1979) On the analysis of geodetic networks for the determination of the incremental strain tensor. Survey Review 25: 146-162
Featherstone, W. E.; (1997) A comparison of existing co-orninate transformation models and parameters in Australia. Cartography 26: 13-25
Grafarend, E.; (1985) Variance-Covariance component estimation; theoretical results and geodetic applications. Statistics & Decisions, Supplement Issue No. 2: 407-441
Grafarend, E.; Sanso, F.; (1985) Optimization and design of geodetic networks. Springer-Verlag, Berlin-Heidelberg-New York-Tokyo
Grafarend, E.; Schaffrin, B.; (1974) Unbiased Freenet adjustment. Survey Review 22: 200-218
Grafarend, E.; Schaffrin, B.; (1983) Ausgleichungsrechnung in Linearen Modellen. B. I. Wissenschaftsverlag, Mannheim
Grafarend, E.; Schaffrin, B.; (1989) The geometry of nonlinear adjustment – the planar trisection problem. In: E. Kejlso, K. Poder and C. C. Tscherning. Festschrift to T. Krarup, pp. 149-172, Denmark
Grafarend, E.; Schaffrin, B.; (1991) The planar trisection problem and the impact of curvature on non-linear least-squares estimation. Computational statistics & data analysis 12: 187-199
Grafarend, E.; Syffus, R.; (1997) Strip transformation of conformal coordinates of type Gauss-Kruger and UTM. Allgemeine Vermessungs-Nachrichten 104: 184-190
Grafarend, E.; Syffus, R.; (1998) Transformation of conformal coordinates of type Gauss-Krüger or UTM from local datum (regional, national, European) to global datum (WGS 84) part 1: The transformation equations. Allgemeine Vermessungs-Nachrichten 105: 134-141
Grafarend, E.; Okeke, F.; (1998) Transformation of conformal coordinates of type Mercator from global datum (WGS 84) to local datum (regional, national). Marine Geodesy 21: 169-180
Grafarend, E.; Awange, J. L.; (2000) Determination of vertical deflections by GPS/LPS measurements. Zeitschrift für Vermessungswesen 125: 279-288
Grafarend, E.; Krumm, F.; Okeke, F.; (1995) Curvilinear geodetic datum transformation. Zeitschrift für Vermessungswesen 120: 334-350
Grafarend, E.; Hendricks, A.; Gilbert, A.; (2000) Transformation of conformal coordinates of type Gauss-Kruger or UTM from a local datum (Regional, National, European) to a global datum (WGS 84, ITRF 96). Allgemeine Vermessungs-Nachrichten 107: 218-222
Harvey, B. R.; (1986) Transformation of 3D coordinates. The Australian Surveyor 33: 105-125
Jacobi C. G. I.; (1841) Deformatione et proprietatibus determinantum. Crelle's Journal für die reine und angewandte Mathematik, Bd. 22
Kampmann, G.; (1996) New adjustment techniques for the determination of transformation parameters for Cadastral and Engineering purposes. Geomatica 50: 27-34
Koch, K. R.; (1999) Parameter estimation and hypothesis testing in linear models. 2nd Edition, Springer-Verlag, Berlin-Heidelberg
Meissl, P.; (1982) Least squares adjustment. A modern approach, Mitteilungen des geodätischen Instituts der Technischen Universität Craz, Folge 43
Mittermayer, E.; (1972) A generalization of least squares adjustment of free networks. Bull. Geod. No. 104: 139-155
Nitschke, M.; Knickmeyer, E. H.; (2000) Rotation parameters – a survey of techniques. Journal of Surveying Engineering 126: 83-105
Okeke, F. I.; (1998) The curvilinear datum transformation model. DGK, Reihe C, Heft Nr. 481
Perelmuter, A.; (1979) Adjustment of free networks. Bull. Geod. 53: 291-295
Pistone, G.; Wynn, H. P.; (1996) Generalized confounding with Gröbner bases. Biometrika 83: 112-119
Rao C. R.; (1967) Least squares theory using an estimated dispersion matrix and its application to measurement of signals. Procedure of the Fifth Barkeley Symposium, Barkeley
Rao C. R.; (1971) Estimation of variance and covariance components – MINQUE theory. Journal of Multivariate Analysis 1: 257-275
Rao C. R.; (1973) Representation of the best linear unbiased estimators in the Gauss-Markov model with singular dispersion matrix. Journal of multivariate analysis 3: 276-292
Rao C. R.; (1978) Choice of the best linear estimators in the Gauss-Markov model with singular dispersion matrix. Comm. Stat. Theory Meth. A7 (13): 1199-1208
Rao C. R.; Kleffe, J.; (1979) Variance and covariance components estimation and applications. Technical Report No. 181, Ohio State University, Dept. of Statistics, Columbus, Ohio
Schaffrin, B.; (1983) Varianz-Kovarianz-Komponenten-Schätzung bei der Ausgleichung heterogener Wiederholungsmessungen. DGK, Reihe C, Heft Nr. 282
Shut, G. H.; (1958/59) Construction of orthogonal matrices and their application in analytical Photogrammetrie. Photogrammetria XV: 149-162
Sturmfels, B.; (1996) Gröbner bases and convex polytopes. American Mathematical Society, Providence
Thompson, E. H.; (1959a) A method for the construction of orthogonal matrices. Photogrammetria III: 55-59
Thompson, E. H.; (1959b) An exact linear solution of the absolute orientation. Photogrammetria XV: 163-179
Vanicek, P.; Steeves, R. R.; (1996) Transformation of coordinates between two horizontal geodetic datums. Journal of Geodesy 70: 740-745
Welsch, W. M.; (1993) A general 7-parameter transformation for the combination, comparison and accuracy control of the terrestrial and satellite network observations. Manuscripta Geodaetica 18: 295-305
Wolfrum, O.; (1992) Merging terestrial and satellite networks by a ten-parameter transformation model. Manuscripta Geodaetica 17: 210-214
Yang, Y.; (1999) Robust estimation of geodetic datum transformation. Journal of Geodesy 73: 268-274

Joseph L. Awange, Erik W. Grafarend
Department of Geodesy and GeoInformatics, University of Stuttgart
Geschwister-Scholl-Str. 24
D-70174 Stuttgart

Abstract

We present here the *Gauss-Jacobi combinatorial* algorithm to solve in a closed form the overdetermined problem of *7-parameters datum transformation*. From the nine *7-parameters datum transformation* **equations, 36 minimum combinatorial equations, each comprising 7 equations are formed and solved using the *Groebner basis* algorithm in the first step. With each minimal combinatorial subset yielding 7 elements of the solution set, a total of $\{7 \times 36 = 252\}$ solutions are formed. The *minimum combinatorial solutions* are reduced to their final adjusted values in step two by means of their weighted mean via the Error propagation law/variance-covariance propagation. The advantage is that the *Groebner basis* algorithm which is already implemented in algebraic softwares such as Mathematica and Maple does not require approximate values as is always the case with traditional procedures. The procedure makes it possible for the stochasticity of both coordinate systems involved to be taken into account.**

Key words

Gauss-Jacobi combinatorial algorithm. 7-parameter datum transformation. Gröbner basis.

DEUMLICH

Entwicklungsprobleme der Geodäsie und Kartografie

In der Geodäsie und Kartografie Russlands wird die technologische Umrüstung als eine der wichtigsten gegenwärtigen Aufgaben bezeichnet. Ihre Hauptrichtungen sind

- die Nutzung globaler Navigationssysteme für die geodätische Erfassung des Landes
- der Übergang der geodätisch-topografischen und der kartografischen Produktion zu computergestützten Technologien.

Hierfür dienen entsprechende Forschungen, insbesondere zur Nutzung des russischen Satellitensystems GLONASS, da Entwicklungen zur Verwendung des amerikanischen GPS-Systems wegen dessen Geheimhaltung nicht in Frage kommen.

Für GLONASS muss ein System für genaue Bestimmungen sowohl der Bord- als auch der a-posteriori-Ephemeriden geschaffen werden. Letztere sind besonders wichtig zur Koordinatenbestimmung höchster Genauigkeit. Zur Koordinierung der Ephemeridenbestimmung wird erwogen, ein methodisches und ein Rechenzentrum zu bilden sowie entsprechende Programme und Verbindungslinien zwischen den Messpunkten auszuarbeiten. Für die Programme sind Methoden sowie einfache präzisierte Systeme der Erdparameter anhand von Satelliten-, astronomisch-geodätischen und gravimetrischen Daten, weiterhin Methoden der Präzisionsorientierung der terrestrischen Koordinatensysteme auf Grund von radiointerferometrischen und satellitengeodätischen Messungen zu entwickeln. All das ermöglicht die Herstellung einer geodätischen Grundlage, die auch für geodynamische Forschungen geeignet ist.

Für viele Nutzer der Informationen sind Koordinaten mit einer Genauigkeit von einigen Zentimetern, die unmittelbar im Gelände mit Satellitenempfängern bestimmt werden, besonders wichtig. Sie sind im absoluten Regime mit den vorhandenen Navigationssystemen kaum zu erhalten. Bekanntlich ist die hochgenaue Koordinatenbestimmung aber mit Satellitendifferentialstationen möglich. Problematisch ist dabei der ökonomische Effekt dieser Methode bei relativ geringen Abständen der Differentialstationen von den zu bestimmenden Punkten sowie die geringe Anzahl der Nutzer. Daher wird die Entwicklung von Theorien und Methoden für diese Aufgaben als sehr effektiv angesehen.

Die nächsten Aufgaben der Geodäsie in Russland werden durch die Einführung des Koordinatensystems SK-95 bestimmt. Erfahrungen zeigen, dass hierfür im Jahre 2002 sowohl die entsprechenden Fachkräfte als auch die Finanzierung zu sichern sind.

Auf dem Gebiet der Topografie sind Forschungen zur Verminderung der Feldarbeiten vordringlich. Hierfür wurde eine durchgehende Technologie vom digitalen Bild bis zum Erhalt des Herausgabeoriginals der topografischen Karte entwickelt, die zu vervollkommnen ist. Zur Minimierung der Feldarbeiten ist die Automatisierung der Interpretation und die umfangreiche Anwendung der Luftbilder mit der Koordinatenbestimmung der Projektionszentren erforderlich – eine Aufgabe der nächsten Jahre.

Aus: O problemach razvitija otrasli. Von Makarenko, N.L. – Geodez. i Kartogr., Moskva (2002) 5, S. 1–3

DEUMLICH