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Sensitive control of high-speed-railway tracks

Part I: Local representation of the clothoid

0 Introduction

High-speed-railways also called bullet trains or Transrapid need *extreme reliable control systems* (*"extasy"*) to avoid catastrophic events. Here we consequently focus on two sensitive problems related to *extasy*:

- a local high resolution representation of the track design (clothoid, circle, straight line) in UTM *map matching coordinates* is given.
- the *Mixed Model* (*universal Kriging*) is used to discriminate measurement errors from track displacements.

In this first part we develop the local high resolution representation of the *clothoid* (special case: circle, straight line) which is needed for creating an *expert base* of *extasy*.

1 Local representation of the clothoid

At first we are deriving the differential equation which generates the special curve *clothoid*. The initial value problem of such a differential equation is solved in terms of the *Fresnel integrals*. *Secondly* we succeed to solve the *Fresnel integrals* by a power series expansion the azimuth functions (sin α(s), cos α(s)) relative to the *initial curvature* κ_0 of the *clothoid*. In this way the coordinate functions $x-x_0 = f\,(\alpha_0,\ \kappa_0,\ s-s_0)$ and $y-y_0 = g\,(\alpha_0,\ \kappa_0,\ s-s_0)$ are derived, namely for (x, y) as conformal coordinates of *Gauss-Krueger* or *UTM type*. *Thirdly* we take advantage of *univariate series inversion* in order to derive the *clothoid function* $y-y_0 = h(x-x_0;\ \alpha_0,\ \kappa_0)$. As special cases the straight line and the circle are included. *Fourthly* we present case studies for the local representation of the *clothoid* for various degrees of approximations.

1-1 Initial value problem of the clothoid

In the *Gauss-Krueger or UTM plane* we consider a planar curve $\mathbf{x}(s)$ which is parameterized by its *arc length s*. For a local representation of such a curve we introduce the *orthonormal Frenet frame* $\{\mathbf{f}_1, \mathbf{f}_2\}$ which *moves* with respect to the *orthonormal Euclid frame* $\{\mathbf{e}_1, \mathbf{e}_2\,|\mathbf{0}\}$ fixed to the origin 0. By means of *Gram-Schmidt orthonormalization* a constructive set-up of such a moving frame is

$$\mathbf{f}_1 = \mathbf{x}'(s), \quad \mathbf{f}_2 = \frac{\mathbf{x}'' - \langle \mathbf{x}''|\mathbf{x}'\rangle \mathbf{x}'}{\|\mathbf{x}'' - \langle \mathbf{x}''|\mathbf{x}'\rangle \mathbf{x}'\|}. \tag{1.1}$$

Here $\langle\bullet|\bullet\rangle$ denotes the standard *Euclidean scalar product* as well as $\|\bullet\|$ the standard *Euclidean norm* (l_2 – *norm*). $\boldsymbol{\mu} := \mathbf{f}_1$ is called normalized *tangent vector*, $\mathbf{v} := \mathbf{f}_2$ normalized *normal vector* of the planar curve $\mathbf{x}(s)$. The moving frame $\{\mathbf{f}_1(s), \mathbf{f}_2(s)\}$ is related to the fixed frame $\{\mathbf{e}_1, \mathbf{e}_2|\mathbf{0}\}$ by

$$\mathbf{f}^* = [\mathbf{f}_1, \mathbf{f}_2] = [\mathbf{e}_1, \mathbf{e}_2]\mathbf{R}^* = \mathbf{e}\mathbf{R}^* \tag{1.2}$$

Where $\mathbf{R}$ is the set $\mathbf{R} \in SO(2)$ of orthonormal matrices, namely $\mathbf{R} \in \{\mathbf{R} \in \mathbb{R}^{2\times 2} | \mathbf{R}\mathbf{R}^* = \mathbf{I}_2, |\mathbf{R}| = +1\}$. $\mathbf{R}^*$ denotes the transpose of $\mathbf{R}$.

$$\mathbf{R} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}, \qquad \alpha(s) \tag{1.3}$$

is the representation of the rotation matrix in terms of the polar coordinate α. As an angle α describes the *circular motion* of the tangent vector $\boldsymbol{\mu}$ as well as the normal vector $\boldsymbol{\nu}$.

The *Frenet equations* are the derivational equations $\mathbf{f}' = \mathbf{e}(\mathbf{R}')^* = \mathbf{f}\mathbf{R}(\mathbf{R}')^* = \mathbf{f}\boldsymbol{\Omega}^*$ where $\boldsymbol{\Omega} := \mathbf{R}'\mathbf{R}^*$ denotes the *Cartan matrix*, as an antisymmetric matrix subject to the so(2) algebra. $\boldsymbol{\Omega} \in \mathbb{R}^{2\times 2}$ as an *antisymmetric* matrix is structured by only one nonvanishing element, namely $\omega_{12} = \kappa(s)$, called *curvature* of the planar curve.

$$\boldsymbol{\Omega} = \begin{bmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{bmatrix}, \qquad \begin{aligned} \mathbf{f}_1'(s) &= \kappa(s)\mathbf{f}_2 \\ \mathbf{f}_2'(s) &= -\kappa(s)\mathbf{f}_1 \end{aligned} \tag{1.4}$$

We are going to derive the angular representation of curvature $\kappa(s)$. An explicit writing of the identity $\mathbf{f} = \mathbf{e}\mathbf{R}^*$ is

$$\begin{aligned} \mathbf{f}_1 &= \mathbf{e}_1\cos\alpha + \mathbf{e}_2\sin\alpha \\ \mathbf{f}_2 &= -\mathbf{e}_1\sin\alpha + \mathbf{e}_2\cos\alpha \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \mathbf{f}_1\cos\alpha - \mathbf{f}_2\sin\alpha &= \mathbf{e}_1 \\ \mathbf{f}_1\sin\alpha + \mathbf{f}_2\cos\alpha &= \mathbf{e}_2 \end{aligned} \tag{1.5}$$

differentiated to

$$\begin{aligned} \mathbf{f}_1' &= -\mathbf{e}_1\alpha'\sin\alpha + \mathbf{e}_2\alpha'\cos\alpha = \mathbf{f}_2\alpha' \\ \mathbf{f}_2' &= -\mathbf{e}_1\alpha'\cos\alpha - \mathbf{e}_2\alpha'\sin\alpha = -\mathbf{f}_1\alpha'\,. \end{aligned} \tag{1.6}$$

Indeed prime differentiation refers to differentiation with respect to *arc length s*. The final result of the differentiation we collect in

Corollary 1.1 (angular representation of curvature):

$$\kappa(s) = \alpha'(s) \tag{1.7}$$

For the proof we just have to identify $\omega_{12} = \kappa(s)$ within $\mathbf{f}_1'$ and $\mathbf{f}_2'$, respectively. A *clothoid* may now be defined as such a curve whose product of curvature radius $r(s) := 1/\kappa(s)$ and its *arc length* s is a positive constant, namely $rs = a^2$. Conversely we take advantage of

Definition 1.2 ("clothoid"):

A planar curve is called clothoid $\mathbb{C}$ *if its curvature is positively proportional to the arc length* s, in particular

$$\kappa(s) = s/a^2 \tag{1.8}$$

The circular motion of the tangent vector $\boldsymbol{\mu}(s)$ as well as normal vector $\boldsymbol{\nu}(s)$ of the *clothoid* can be conveniently described by solving the *initial value problem* $\alpha' = \kappa(s) = s/a^2$, $\alpha_0 = \alpha(s_0)$, $\alpha'_0 = \alpha'(s_0)$, solved by

Corollary 1.3 (circular motion of the tangent vector and the normal vector of the clothoid):

The initial value problem $(i)\ \alpha' = s/a^2,\ s \in \mathbb{R}^+$, $(ii)\ \alpha_0 = \alpha(s_0), \alpha'_0 = \alpha'(s_0)$ *is solved by*

$$\alpha(s) = \alpha_0 + \alpha'_0(s - s_0) + \frac{1}{2!}\alpha''_0(s - s_0)^2 = \alpha_0 + \kappa_0(s - s_0) + \frac{1}{2!}\kappa'_0(s - s_0)^2 \tag{1.9}$$

subject to

$$\kappa_0 = \kappa(s_0) = \frac{s_0}{a^2}, \quad \kappa'_0 = \kappa'(s_0) = \frac{1}{a^2} \tag{1.10}$$

For the proof we have to find the general solution of the homogeneous equation $\alpha' = 0$ and a particular solution of the inhomogeneous equation $\alpha' = s/a^2$. *First* the general solution of the *homogeneous equation* is

$$\alpha(s) = \alpha_0 + \alpha'_0(s - s_0) \quad or \quad \alpha(s) = \alpha_0 + \frac{s_0}{a^2}(s - s_0) \tag{1.11}$$

Second a particular solution of the inhomogeneous equation $\alpha' = s/a^2$ is based upon the integral

$$\alpha(s) = \int_{s_0}^{s} \kappa(s)ds = \frac{1}{a^2}\int_{s_0}^{s} sds = \frac{1}{2a^2}(s - s_0)^2 \tag{1.12}$$

The superposition of the general solution of the homogeneous equation and of the particular solution of the inhomogeneous equation leads directly to the result of *Corollary 1.3.*

The *clothoid* $\mathbb{C} \subset \mathbb{R}^2$ isometrically embedded in $\mathbb{R}^2$ has finally to be constructed from its curvature $\kappa(s) = \langle \mathbf{x}'' | \boldsymbol{\nu}(s) \rangle$ indeed a problem of *global differential geometry*. Since in the *first step* we have already characterized the circular motion of its tangent vector as well as its normal vector, in the *second step* we shall concentrate on its embedding function $\mathbf{x}(s)$. The tangent vector $\mathbf{x}'(s)$ at the point s enjoys a particular form in the ambient space $\mathbb{R}^2$, namely

$$\mathbf{x}'(s) = \mathbf{e}_1 x'(s) + \mathbf{e}_2 y'(s) = \mathbf{e}_1 \cos\alpha(s) + \mathbf{e}_2 \sin\alpha(s) \tag{1.13}$$

Here, for the first time Cartesian coordinates (x, y) covering $\mathbb{R}^2$ appear. They can be thought as conformal coordinates of type *Gauss-Krueger* or *UTM* with respect to an *International Reference Ellipsoid*, e.g. WGS 80. In comparing the left and right representation of the tangent vector $\mathbf{x}'(s)$ we are led to the system of differential equations of first order which govern the computation of (x, y) coordinates from the orientation parameter α of the tangent map $\boldsymbol{\mu}(s) \in \mathbb{T}_s\mathbb{M} \to \mathbb{S}^1$, namely

$$x' = \cos\alpha(s), \quad y' = \sin\alpha(s) \tag{1.14}$$

1-2 Series expansion of the Fresnel integrals

If we integrate directly these differential equations, in particular with respect to $\alpha(s)$ representing via *Corollary 1.3* the clothoid $\mathbb{C}$, we are led to the famous *Fresnel integrals*. Such an approach is not suited here since *first* Fresnel integrals are only tabulated and *second* they are too inflexible to account for those parts of the clothoid $\mathbb{C}$ which are by purpose interrupted by *circles and straight lines*. Instead a local representation is searched for which can be easily adjusted to curve sections of type circle and/or straight line. Our result of integration (x', y'), respectively, is collected in

Lemma 1.4 (local representation of the clothoid $\mathbb{C}$):

Given the system of ordinary differential equations of first order

$$x' = \cos\alpha(s), \quad y' = \sin\alpha(s), \tag{1.15}$$

then the solution of its initial value problem $(x_0, y_0) = (x(s_0), y(s_0))$ is

$$x - x_0 = \int\limits_{s_0}^{s} \cos\alpha(s)ds, \qquad y - y_0 = \int\limits_{s_0}^{s} \sin\alpha(s)ds \tag{1.16}$$

$$\begin{aligned}\Delta x := x - x_0 = {} & \cos\alpha_0 \Delta s - \frac{1}{2}\kappa_o \sin\alpha_0 \Delta s^2 - \frac{1}{6}\cos\alpha_0(\kappa_o^2 + \kappa_o' \tan\alpha_0)\Delta s^3 + \\ & + \frac{1}{24}\kappa_o \cos\alpha_0(\kappa_o^2 \tan\alpha_0 - 3\kappa_o')\Delta s^4 + \frac{1}{120}\cos\alpha_0(\kappa_o^4 + 6\kappa_o^2\kappa_o' \tan\alpha_0 - 3\kappa_o'^2)\Delta s^5 - \\ & - \frac{1}{720}\kappa_o \cos\alpha_0(\kappa_o^4 \tan\alpha_0 - 10\kappa_o^2\kappa_o' - 15\kappa_o'^2 \tan\alpha_0)\Delta s^6 + 0(\Delta s^7)\end{aligned} \tag{1.17}$$

$$\begin{aligned}\Delta y := y - y_0 = {} & \sin\alpha_0 \Delta s - \frac{1}{2}\kappa_o \cos\alpha_0 \Delta s^2 - \frac{1}{6}\cos\alpha_0(\kappa_o^2 \tan\alpha_0 - \kappa_o')\Delta s^3 - \\ & - \frac{1}{24}\kappa_o \cos\alpha_0(\kappa_o^2 + 3\kappa_o' \tan\alpha_0)\Delta s^4 + \frac{1}{120}\cos\alpha_0(\kappa_o^4 \tan\alpha_0 - 6\kappa_o^2\kappa_o' - 3\kappa_o'^2 \tan\alpha_0)\Delta s^5 + \\ & + \frac{1}{720}\kappa_o \cos\alpha_0(\kappa_o^4 + 10\kappa_o^2\kappa_o' \tan\alpha_0 - 15\kappa_o'^2)\Delta s^6 + 0(\Delta s^7)\end{aligned} \tag{1.18}$$

accurate up to order seven in $\Delta s := s - s_0$. The initial curvatures $\kappa_o = \kappa(s_0)$ are classified as

(i) $\kappa_o = 0$ (straight line $\mathbb{L}^1$)
(ii) $\kappa_o = 1/r_0$ (r_0 radius of circle $\mathbb{S}^1_{r_0}$)
(iii) $\kappa_o = s_0/a^2$ (a^2 positive parameter of clothoid $\mathbb{C}_{a^2}$)

For the proof we depart from the integrals

$$x - x_0 = \int\limits_{s_0}^{s} \cos\alpha(s)ds = \int\limits_{s_0}^{s} \cos\left[\alpha_0 + \kappa_o(s - s_0) + \frac{1}{2}\kappa_o'(s - s_0)^2\right] d(s - s_0), \tag{1.19}$$

$$y - y_0 = \int\limits_{s_0}^{s} \sin\alpha(s)ds = \int\limits_{s_0}^{s} \sin\left[\alpha_0 + \kappa_o(s - s_0) + \frac{1}{2}\kappa_o'(s - s_0)^2\right] d(s - s_0). \tag{1.20}$$

subject to

$$\alpha(s) = \alpha_0 + \Delta\alpha, \quad \Delta\alpha := \kappa_0(s - s_0) + \frac{1}{2}\kappa_0'(s - s_0)^2 \quad (1.21)$$

Table 1.1: Power series of $\cos(\alpha_0 + \Delta\alpha), \sin(\alpha_0 + \Delta\alpha)$

$$\begin{aligned}\cos(\alpha_0 + \Delta\alpha) = {} & \cos\alpha_0 - \frac{1}{1!}\sin\alpha_0 \Delta\alpha - \frac{1}{2!}\cos\alpha_0 \Delta\alpha^2 + \frac{1}{3!}\sin\alpha_0 \Delta\alpha^3 + \\ & + \frac{1}{4!}\cos\alpha_0 \Delta\alpha^4 - \frac{1}{5!}\sin\alpha_0 \Delta\alpha^5 + 0_c(\Delta\alpha^6)\end{aligned}$$

$$\begin{aligned}\sin(\alpha_0 + \Delta\alpha) = {} & \sin\alpha_0 - \frac{1}{1!}\cos\alpha_0 \Delta\alpha - \frac{1}{2!}\sin\alpha_0 \Delta\alpha^2 - \frac{1}{3!}\cos\alpha_0 \Delta\alpha^3 + \\ & + \frac{1}{4!}\sin\alpha_0 \Delta\alpha^4 + \frac{1}{5!}\cos\alpha_0 \Delta\alpha^5 + 0_s(\Delta\alpha^6)\end{aligned}$$

Table 1.2: Power series of the incremental orientation parameter $\Delta\alpha$

$$\Delta\alpha = \kappa_0 \Delta s + \kappa_0' \Delta s^2 / 2$$

$$\Delta\alpha^2 = \kappa_o^2 \Delta s^2 + \kappa_o \kappa_0' \Delta s^3 + \kappa_0'^2 \Delta s^4 / 4$$

$$\Delta\alpha^3 = \kappa_o^3 \Delta s^3 + 3\kappa_0^2 \kappa_o' \Delta s^4 / 2 + 3\kappa\kappa_0'^2 \Delta s^5 / 4 + \kappa_0'^3 \Delta s^6 / 8$$

$$\Delta\alpha^4 = \kappa_o^4 \Delta s^4 + 3\kappa_0^3 \kappa_o' \Delta s^5 / 2 + 3\kappa_o^2 \kappa_0'^2 \Delta s^6 / 2 + \kappa_0'^4 \Delta s^8 / 16$$

$$\Delta\alpha^5 = \kappa_o^5 \Delta s^5 + 5\kappa_0^4 \kappa_o' \Delta s^6 / 2 + 5\kappa_o^3 \kappa_0'^2 \Delta s^7 / 2 + 5\kappa_o^2 \kappa_0'^3 \Delta s^8 / 4 + 5\kappa_o \kappa_0'^4 \Delta s^9 / 16 + \kappa_0'^5 \Delta s^{10} / 32$$

Table 1.3: Power series expressions of *Fresnel integrals*

$$x - x_0 = \int_{s_0}^{s} \left[\cos\alpha_0 - \frac{1}{1!}\sin\alpha_0 \Delta\alpha - \frac{1}{2!}\cos\alpha_0 \Delta\alpha^2 + \frac{1}{3!}\sin\alpha_0 \Delta\alpha^3 + \right.$$
$$\left. + \frac{1}{4!}\cos\alpha_0 \Delta\alpha^4 - \frac{1}{5!}\sin\alpha_0 \Delta\alpha^5 + 0_c(\Delta\alpha^6) \right] d(s - s_0)$$

$$y - y_0 = \int_{s_0}^{s} \left[\sin\alpha_0 + \frac{1}{1!}\cos\alpha_0 \Delta\alpha - \frac{1}{2!}\sin\alpha_0 \Delta\alpha^2 - \frac{1}{3!}\sin\alpha_0 \Delta\alpha^3 + \right.$$
$$\left. + \frac{1}{4!}\sin\alpha_0 \Delta\alpha^4 + \frac{1}{5!}\cos\alpha_0 \Delta\alpha^5 + 0_s(\Delta\alpha^6) \right] d(s - s_0)$$

1st integrals

$$\int_{s_0}^{s} \cos\alpha_0 ds = \cos\alpha_0 (s - s_0), \qquad \int_{s_0}^{s} \sin\alpha_0 ds = \sin\alpha_0 (s - s_0)$$

2nd integrals

$$\int_{s_0}^{s} \sin\alpha_0 \Delta\alpha ds = \sin\alpha_0 \left[\kappa_0 \int_0^{s-s_0} \Delta s d\Delta s + \kappa_0' \int_0^{s-s_0} \Delta s^2 d\Delta s / 2 \right]$$

$$\int_{s_0}^{s} \cos\alpha_0 \Delta\alpha ds = \cos\alpha_0 \left[\kappa_0 \int_0^{s-s_0} \Delta s d\Delta s + \kappa_0' \int_0^{s-s_0} \Delta s^2 d\Delta s / 2 \right]$$

$$-\frac{1}{1!}\int_{s_0}^{s} \sin\alpha_0 \Delta\alpha ds = -\kappa_o \sin\alpha_0 (s - s_0)^2 / 2 - \kappa_o' \sin\alpha_0 (s - s_0)^3 / 6$$

$$+\frac{1}{1!}\int_{s_0}^{s} \cos\alpha_0 \Delta\alpha ds = \kappa_o \cos\alpha_0 (s - s_0)^2 / 2 + \kappa_o' \cos\alpha_0 (s - s_0)^3 / 6$$

3rd integrals

$$\int_{s_0}^{s} \cos\alpha_0 \Delta\alpha^2 ds = \cos\alpha_0 \left[\kappa_o^2 \int_0^{s-s_0} \Delta s^2 d\Delta s + \kappa_o \kappa_0' \int_0^{s-s_0} \Delta s^3 d\Delta s + \kappa_0'^2 \int_0^{s-s_0} \Delta s^4 d\Delta s / 4 \right]$$

$$\int_{s_0}^{s} \sin\alpha_0 \Delta\alpha^2 ds = \sin\alpha_0 \left[\kappa_o^2 \int_0^{s-s_0} \Delta s^2 d\Delta s + \kappa_o \kappa_0' \int_0^{s-s_0} \Delta s^3 d\Delta s + \kappa_0'^2 \int_0^{s-s_0} \Delta s^4 d\Delta s / 4 \right]$$

$$-\frac{1}{2!}\int_{s_0}^{s} \cos\alpha_0 \Delta\alpha^2 ds = -\kappa_o^2 \cos\alpha_0 (s-s_0)^3 / 6 - \kappa_o \kappa_o' \cos\alpha_0 (s-s_0)^4 / 8$$

$$-\frac{1}{2!}\int_{s_0}^{s} \sin\alpha_0 \Delta\alpha^2 ds = -\kappa_o^2 \sin\alpha_0 (s-s_0)^3 / 6 - \kappa_o \kappa_o' \sin\alpha_0 (s-s_0)^4 / 8$$

4th integrals

$$\int_{s_0}^{s} \sin\alpha_0 \Delta\alpha^3 ds = \sin\alpha_0 \left[\kappa_o^3 \int_0^{s-s_0} \Delta s^3 d\Delta s + 3\kappa_o^2 \kappa_0' \int_0^{s-s_0} \Delta s^4 d\Delta s / 2 + 3\kappa_o \kappa_0'^2 \int_0^{s-s_0} \Delta s^5 d\Delta s / 4 + \kappa_0'^3 \int_0^{s-s_0} \Delta s^6 d\Delta s / 8 \right]$$

$$\int_{s_0}^{s} \cos\alpha_0 \Delta\alpha^3 ds = \cos\alpha_0 \left[\kappa_o^3 \int_0^{s-s_0} \Delta s^3 d\Delta s + 3\kappa_o^2 \kappa_0' \int_0^{s-s_0} \Delta s^4 d\Delta s / 2 + 3\kappa_o \kappa_0'^2 \int_0^{s-s_0} \Delta s^5 d\Delta s / 4 + \kappa_0'^3 \int_0^{s-s_0} \Delta s^6 d\Delta s / 8 \right]$$

$$+\frac{1}{3!}\int_{s_0}^{s} \sin\alpha_0 \Delta\alpha^3 ds = \kappa_o^3 \sin\alpha_0 (s-s_0)^4 / 24 + \kappa_o^2 \kappa_o' \sin\alpha_0 (s-s_0)^5 / 4 +$$

$$+\kappa_o \kappa_o'^2 \sin\alpha_0 (s-s_0)^6 / 48 + \kappa_o'^3 \cos\alpha_0 (s-s_0)^7 / 336$$

$$-\frac{1}{3!}\int_{s_0}^{s} \cos\alpha_0 \Delta\alpha^3 ds = -\kappa_o^3 \cos\alpha_0 (s-s_0)^4 / 24 - \kappa_o^2 \kappa_o' \cos\alpha_0 (s-s_0)^5 / 4 -$$

$$-\kappa_o \kappa_o'^2 \cos\alpha_0 (s-s_0)^6 / 48 - \kappa_o'^3 \cos\alpha_0 (s-s_0)^7 / 336$$

5th integrals

$$\int_{s_0}^{s} \cos\alpha_0 \Delta\alpha^4 ds = \cos\alpha_0 \left[\kappa_o^4 \int_0^{s-s_0} \Delta s^4 d\Delta s + 3\kappa_o^3 \kappa_0' \int_0^{s-s_0} \Delta s^5 d\Delta s / 2 + 3\kappa_o^2 \kappa_0'^2 \int_0^{s-s_0} \Delta s^6 d\Delta s / 2 + \kappa_0'^4 \int_0^{s-s_0} \Delta s^8 d\Delta s / 16 \right]$$

$$\int_{s_0}^{s} \sin\alpha_0 \Delta\alpha^4 ds = \sin\alpha_0 \left[\kappa_o^4 \int_0^{s-s_0} \Delta s^4 d\Delta s + 3\kappa_o^3 \kappa_0' \int_0^{s-s_0} \Delta s^5 d\Delta s / 2 + 3\kappa_o^2 \kappa_0'^2 \int_0^{s-s_0} \Delta s^6 d\Delta s / 2 + \kappa_0'^4 \int_0^{s-s_0} \Delta s^8 d\Delta s / 16 \right]$$

$$\frac{1}{4!}\int_{s_0}^{s} \cos\alpha_0 \Delta\alpha^4 ds = \kappa_o^4 \cos\alpha_0 (s-s_0)^5 / 120 + \kappa_o^3 \kappa_o' \cos\alpha_0 (s-s_0)^6 / 96 +$$

$$+\kappa_o^2 \kappa_o'^2 \cos\alpha_0 (s-s_0)^7 / 112 + \kappa_o'^4 \cos\alpha_0 (s-s_0)^9 / 3456$$

$$\frac{1}{4!}\int_{s_0}^{s} \sin\alpha_0 \Delta\alpha^4 ds = \kappa_o^4 \sin\alpha_0 (s-s_0)^5 / 120 + \kappa_o^3 \kappa_o' \sin\alpha_0 (s-s_0)^6 / 96 +$$

$$+\kappa_o^2 \kappa_o'^2 \sin\alpha_0 (s-s_0)^7 / 112 + \kappa_o'^4 \sin\alpha_0 (s-s_0)^9 / 3456$$

6th integrals

$$\int_{s_0}^{s} \sin\alpha_0 \Delta\alpha^5 ds = \sin\alpha_0 \left[\kappa_o^5 \int_0^{s-s_0} \Delta s^5 d\Delta s + 5\kappa_o^4 \kappa_0' \int_0^{s-s_0} \Delta s^6 d\Delta s / 2 + 5\kappa_o^3 \kappa_0'^2 \int_0^{s-s_0} \Delta s^7 d\Delta s / 2 + \right.$$

$$+5\kappa_o^2\kappa_0'^3\int_0^{s-s_0}\Delta s^8 d\Delta s/4+5\kappa_o\kappa_0'^4\int_0^{s-s_0}\Delta s^9 d\Delta s/16+\kappa_0'^5\int_0^{s-s_0}\Delta s^{10} d\Delta s/32\Bigg]$$

$$\int_{s_0}^{s}\cos\alpha_0\Delta\alpha^5 ds=\cos\alpha_0\Bigg[\kappa_o^5\int_0^{s-s_0}\Delta s^5 d\Delta s+5\kappa_o^4\kappa_0'\int_0^{s-s_0}\Delta s^6 d\Delta s/2+5\kappa_o^3\kappa_0'^2\int_0^{s-s_0}\Delta s^7 d\Delta s/2+$$

$$+5\kappa_o^2\kappa_0'^3\int_0^{s-s_0}\Delta s^8 d\Delta s/4+5\kappa_o\kappa_0'^4\int_0^{s-s_0}\Delta s^9 d\Delta s/16+\kappa_0'^5\int_0^{s-s_0}\Delta s^{10} d\Delta s/32\Bigg]$$

$$-\frac{1}{5!}\int_{s_0}^{s}\sin\alpha_0\Delta\alpha^5 ds=-\kappa_o^5\sin\alpha_0(s-s_0)^6/720-\kappa_o^4\kappa_o'\sin\alpha_0(s-s_0)^7/336-$$

$$-\kappa_o^3\kappa_o'^2\sin\alpha_0(s-s_0)^8/384-\kappa_o^2\kappa_o'^3\sin\alpha_0(s-s_0)^9/864-$$

$$-\kappa_o\kappa_o'^4\sin\alpha_0(s-s_0)^{10}/3840-\kappa_o'^5\sin\alpha_0(s-s_0)^{11}/42240$$

$$+\frac{1}{5!}\int_{s_0}^{s}\cos\alpha_0\Delta\alpha^5 ds=\kappa_o^5\cos\alpha_0(s-s_0)^6/720+\kappa_o^4\kappa_o'\cos\alpha_0(s-s_0)^7/336+$$

$$+\kappa_o^3\kappa_o'^2\cos\alpha_0(s-s_0)^8/384+\kappa_o^2\kappa_o'^3\cos\alpha_0(s-s_0)^9/864+$$

$$+\kappa_o\kappa_o'^4\cos\alpha_0(s-s_0)^{10}/3840+\kappa_o'^5\cos\alpha_0(s-s_0)^{11}/42240$$

By means of the *uniformly convergent power series* of type $\cos(\alpha_0+\Delta\alpha)$ and $\sin(\alpha_0+\Delta\alpha)$ given in *Table 1.1* and the powers $\Delta\alpha^n = (\alpha_0 + \Delta\alpha)^n$ given in *Table 1.2* up to $n = 5$ we are able to compute the fundamental integrals of Fresnel type outlined in *Table 1.3*. Indeed due to uniform convergence we can apply termwise integration. A fill in of these integrals of order one, two, three, four, five and six into $(x - x_0, y - y_0)$ ordered according to the power of $\Delta s := s - s_0$ leads directly to (1.17) and (1.18).

1-3 Univariate series inversion

In many applications the *clothoid* is uncomfortably parameterized in terms of the *arc length* $s - s_0$ with respect to an initial point $(x(s_0), y(s_0))$. We are going beforehand to derive the parameterization $y - y_0 = f(x - x_0)$ or $\Delta y(\Delta y)$ of the clothoid in terms of the abscissa $\Delta x := x - x_0$. The technique we apply is the *inversion of a univariate homogeneous polynomial of degree n*, e.g. according to E. W. GRAFAREND, T. KRARUP and R. SYFFUS (1996), being outlined in *Table 1.4*. As soon as we have inverted $\Delta x(\Delta s)$ towards $\Delta s(\Delta x)$, we replace $\Delta s(\Delta x)$ within the power series $\Delta y(\Delta s)$. The result is presented in

Lemma 1.5 ($y(x)$ local representation of the clothoid $\mathbb{C}$):

The series inversion of $\Delta x(\Delta s)$ of type (1.17) by *Table 1.4* leads to $\Delta s(\Delta x)$, namely

$$\text{(1.22)}\qquad \Delta s=\sec\alpha_0\Delta x+\frac{1}{2}\kappa_0\sec^2\alpha_0\tan\alpha_0\Delta x^2+\frac{1}{6}\sec^3\alpha_0(\kappa_o^2+3\tan^2\alpha_0+\kappa_o'\tan\alpha_0)\Delta x^3+$$

$$+\frac{1}{24}\kappa_0\sec^4\alpha_0(9\kappa_o^2\tan\alpha_0+15\kappa_o^2\tan^2\alpha_0+3\kappa_o'+10\kappa_o'\tan^2\alpha_0)\Delta x^4+$$

$$+\frac{1}{120}\sec^5\alpha_0(9\kappa_o^4+90\kappa_o^4\tan^2\alpha_0+105\kappa_o^4\tan^4\alpha_0+59\kappa_o^2\kappa_o'\tan\alpha_0+$$

$$+105\kappa_o^2\kappa_o'\tan^3\alpha_0+3\kappa_o'^2+10\kappa_o'^2\tan^2\alpha_0)\Delta x^5+$$

$$+\frac{1}{720}\kappa_o\sec^6\alpha_0(225\kappa_o^4\tan\alpha_0+1050\kappa_o^4\tan^3\alpha_0+$$

$$+1260\kappa_o^2\kappa_0'\tan^4\alpha_0+153\kappa_o'^2\tan\alpha_0+280\kappa_o'^2\tan^3\alpha_0)\Delta x^6+$$

$$+0(\Delta x^7)$$

which is repaced in $\Delta y(\Delta s)$ of type (1.18).

$$
\begin{aligned}
&(1.23)\qquad \Delta y = y - y_0 = \\
&= \Delta x \tan\alpha_0 + \\
&+\frac{1}{2}\Delta x^2 \kappa_0 \sec^3\alpha_0 + \\
&+\frac{1}{6}\Delta x^3 \sec^4\alpha_0 (3\kappa_o^2 \tan\alpha_0 + \kappa_o') + \\
&+\frac{1}{24}\Delta x^4 \kappa_0 \sec^5\alpha_0 \left[3\kappa_o^2 (1 + 5\tan^2\alpha_0) + 10\kappa_o' \tan\alpha_0\right] + \\
&+\frac{1}{120}\Delta x^5 \sec^6\alpha_0 \left[15\kappa_o^4 \tan\alpha_0 (3 + 7\tan^2\alpha_0) + \kappa_o^2\kappa_o'(19 + 105\tan^2\alpha_0) + 10\kappa_o'^2 \tan\alpha_0\right] + \\
&+\frac{1}{720}\Delta x^6 \kappa_o \sec^7\alpha_0 \left[45\kappa_o^4 (1 + 14\tan^2\alpha_0 + 21\tan^4\alpha_0) + \right. \\
&\qquad \left. + 252\kappa_o^2\kappa_o' \tan\alpha_0 (2 + 5\tan^2\alpha_0) + 8\kappa_o'^2 (6 + 35\tan^2\alpha_0)\right] + \\
&+\frac{1}{5040}\Delta x^7 \sec^8\alpha_0 \left[45\kappa_o^6 \tan\alpha_0 (35 + 210\tan^2\alpha_0 + 231\tan^4\alpha_0) + \right. \\
&\qquad + 9\kappa_o^4\kappa_o'(81 + 1218\tan^2\alpha_0 + 1925\tan^4\alpha_0) + 26\kappa_o^2\kappa_o'^2 \tan\alpha_0 (86 + 225\tan^2\alpha_0) + \\
&\qquad \left. + 8\kappa_o'^3 (6 + 35\tan^2\alpha_0)\right] + \\
&+0(\Delta x^8)
\end{aligned}
$$

Table 1.4: Inversion of a univariate homogeneous polynomial of degree *n*, clothoidal application

$$y(x) = a_{11}x + a_{12}x^2 + a_{13}x^3 + a_{14}x^4 + \cdots + a_{1n}x^n$$

$$x(y) = b_{11}y + b_{12}y^2 + b_{13}y^3 + b_{14}y^4 + \cdots + b_{1n}y^n$$

$$b_{11} = a_{11}^{-1}$$

$$b_{12} = -a_{11}^{-3}a_{12}$$

$$b_{13} = -a_{11}^{-1}(a_{12}a_{22}^{-1}a_{23} - a_{13})a_{33}^{-1}$$

$$b_{14} = a_{11}^{-1}\left[a_{12}a_{22}^{-1}(a_{24} - a_{23}a_{33}^{-1}a_{34}) + a_{13}a_{33}^{-1}a_{34} - a_{14}\right]a_{44}^{-1}$$

$$\cdots$$

$$y := x - x_0, \quad x := s - s_0$$

$$:(1.17):$$

$$a_{11} = \cos\alpha_0, \quad a_{12} = -\frac{1}{2}\kappa_0 \sin\alpha_0,$$

$$a_{13} = -\frac{1}{6}\cos\alpha_0(\kappa_o^2 + \kappa_o' \tan\alpha_0), \quad a_{14} = \frac{1}{24}\kappa_o \cos\alpha_0(\kappa_o^2 \tan\alpha_0 - 3\kappa_o'), \cdots$$

Example 1.1 (the circle $\mathbb{S}^1_{r_0}$: $\kappa_o = \frac{1}{r_0}$, $\kappa_o' = 0$):

$$\alpha(s) = \alpha_0 + \int_{s_0}^{s} \kappa(s)ds = \alpha_0 + \kappa_o(s - s_0)$$

$$x - x_0 = \int_{s_0}^{s} \cos\alpha(s)ds = \int_{s_0}^{s} \cos\left[\alpha_0 + \kappa_o(s - s_0)\right]ds = \frac{1}{\kappa_o}\int_{\alpha_0}^{\alpha_0+\kappa_o(s-s_0)} \cos x dx$$

$$y - y_0 = \int_{s_0}^{s} \sin\alpha(s)ds = \int_{s_0}^{s} \sin\left[\alpha_0 + \kappa_o(s - s_0)\right]ds = \frac{1}{\kappa_o}\int_{\alpha_0}^{\alpha_0+\kappa_o(s-s_0)} \sin x dx$$

Corollary 1.6 (embedding of $\mathbb{S}^1_{r_0}$ in $\mathbb{R}^2$):

(1.24) $$x - x_0 = \frac{\sin[\alpha_0 + \kappa_o(s - s_0)] - \sin\alpha_0}{\kappa_o}$$

(1.25) $$y - y_0 = \frac{-\cos[\alpha_0 + \kappa_o(s - s_0)] + \cos\alpha_0}{\kappa_o}$$

$$\cos[\alpha_0 + \kappa_o(s - s_0)] = \sqrt{1 - \sin^2[\alpha_0 + \kappa_o(s - s_0)]^2} = \sqrt{1 - [\kappa_o(x - x_0) + \sin\alpha_0]^2}$$

Corollary 1.7 (embedding of $\mathbb{S}^1_{r_0}$ in $\mathbb{R}^2$):

(1.26) $$y - y_0 = -\sqrt{\frac{1}{\kappa_0^2} - \left[(x - x_0) + \frac{1}{\kappa_o}\sin\alpha_0\right]^2} + \frac{\cos\alpha_0}{\kappa_o}$$

If $\alpha_0 = 0,\ x_0 = 0,\ y_0 = 1/\kappa_0$, *then*

$$y = -\sqrt{r_0^2 - x^2}$$

covers the Southern half circle.

A series expansion of (1.24), (1.25) according to *Table 1.1* leads directly to (1.17) with κ'_0. = 0. Similarly a series expansion of the root in (1.26) approaches (1.23) for κ'_0. = 0.

Example 1.2

(the straight line $\mathbb{L}^1$: $\kappa_o = 0,\ \kappa'_o = 0$)

$$\alpha(s) = \alpha_0$$

$$x - x_0 = \int_{s_0}^{s} \cos\alpha(s)ds = \cos\alpha_0(s - s_0)$$

$$y - y_0 = \int_{s_0}^{s} \sin\alpha(s)ds = \sin\alpha_0(s - s_0)$$

Corollary 1.8 ($\mathbb{L}^1 \subset \mathbb{R}^2$):

(1.27) $$x - x_0 = \cos\alpha_0(s - s_0), \quad y - y_0 = \sin\alpha_0(s - s_0)$$

$$y - y_0 = \tan\alpha_0(x - x_0)$$

Examples 1.1 and *Example 1.2* have clearly documented that the fundamental local representation (1.17), (1.18) as well as (1.23) of the *clothoid* contains the circle and the straight line as special cases. Indeed this has been one target function why we developed (1.17), (1.18), and (1.23).

1-4 Case studies

For the case studies we have collected the initial value data of *Table 1.5* for various degrees of approximation.

Table 1.5: Initial value data for a numerical analysis of the clothoid

Case 1	*Case 1*	*Case 3*
$\kappa_0 = 1/500$	$\kappa_0 = 1/500$	$\kappa_0 = 1/500$
$\kappa_0' = 0$	$\kappa_0' = 0$	$\kappa_0' = 10^{-6}$
$\alpha_0 = 10^\circ$	$\alpha_0 = 10^\circ$	$\alpha_0 = 10^\circ$
$x_0 = y_0 = s_0 = 0$	$x_0 = y_0 = s_0 = 0$	$x_0 = y_0 = s_0 = 0$
(*Figure 1.1*)	(*Figure 1.1*)	(*Figure 1.7*)
$x_0 = y_0 = 200$	$x_0 = y_0 = 200$	$\kappa_0 = 1/500$
$s_0 = 0$	$s_0 = 0$	$\kappa_0' = 10^{-6}$
(*Figure 1.2*)	(*Figure 1.2*)	$\alpha_0 = 45^\circ$
$x_0 = y_0 = 200$	$x_0 = y_0 = 200$	$x_0 = y_0 = 0$
$s_0 = 50$	$s_0 = 50$	$s_0 = 0$
(*Figure 1.3*)	(*Figure 1.3*)	(*Figure 1.8*)

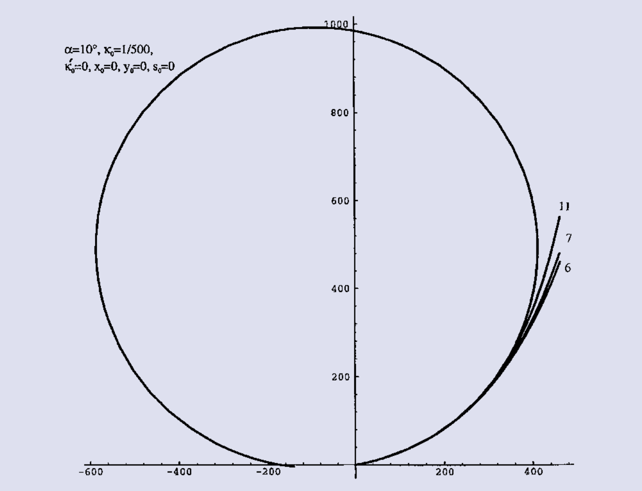

Figure 1.1: Local representation of the circle by (1.23), $\kappa_0 = 1/500$, $\kappa_0' = 0$, $\alpha_0 = 10°$, $x_0 = y_0 = 0$, $s_0 = 0$, various degrees of approximation.

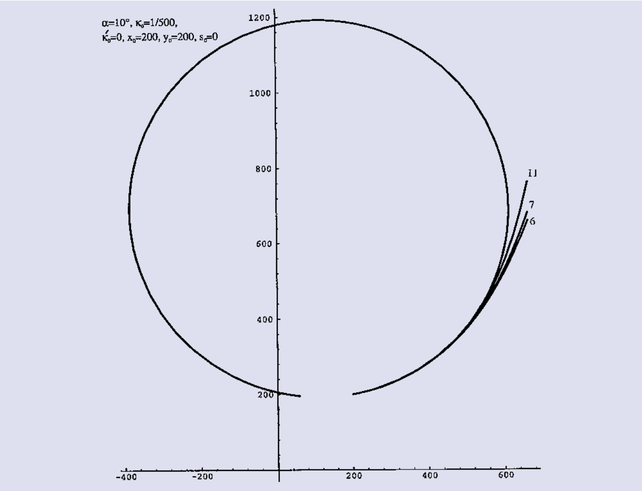

Figure 1.2: Local representation of the circle by (1.23), $\kappa_0 = 1/500$, $\kappa_0' = 0$, $\alpha_0 = 10°$, $x_0 = 200$, $y_0 = 200$, $s_0 = 0$, various degrees of approximation.

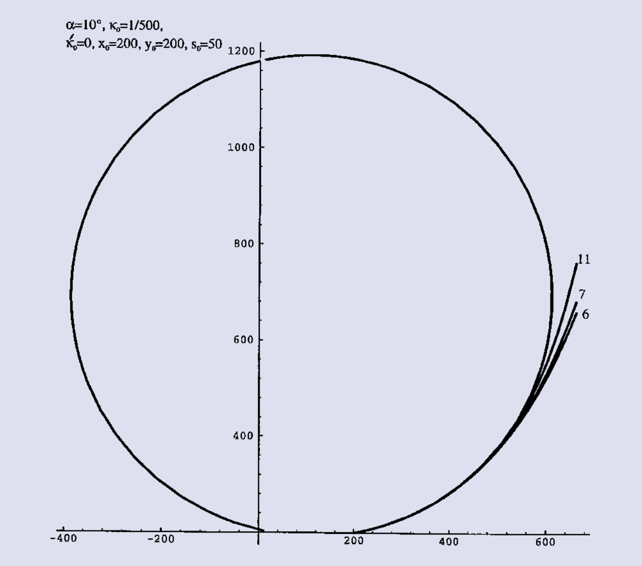

Figure 1.3: Local representation of the circle by (1.23), $\kappa_0 = 1/500$, $\kappa_0' = 0$, $\alpha_0 = 10°$, $x_0 = 200$, $y_0 = 200$, $s_0 = 50$, various degrees of approximation.

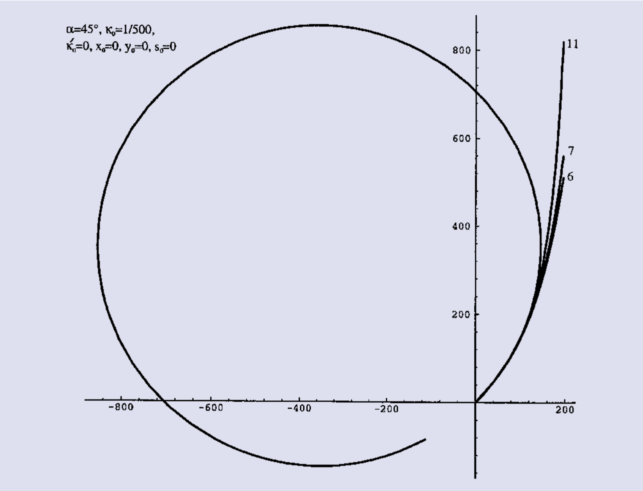

Figure 1.4: Local representation of the circle by (1.23), $\kappa_0 = 1/500$, $\kappa_0' = 0$, $\alpha_0 = 45°$, $x_0 = y_0 = 0$, $s_0 = 0$, various degrees of approximation.

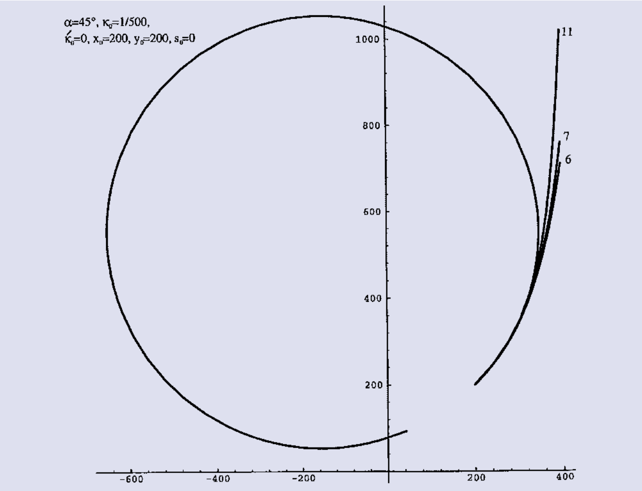

Figure 1.5: Local representation of the circle by (1.23), $\kappa_0 = 1/500$, $\kappa_0' = 0$, $\alpha_0 = 45°$, $x_0 = 200$, $y_0 = 200$, $s_0 = 0$, various degrees of approximation.

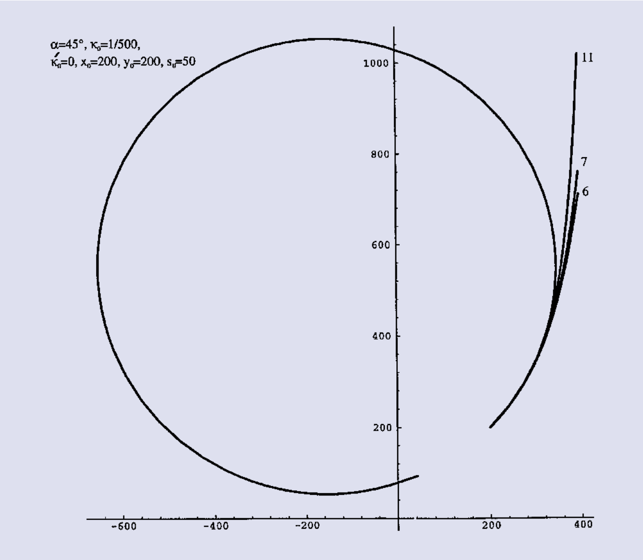

Figure 1.6: Local representation of the circle by (1.23), κ_0 = 1/500, κ_0' = 0, α_0 = 45°, x_0 = 200, y_0 = 200, s_0 = 50, various degrees of approximation.

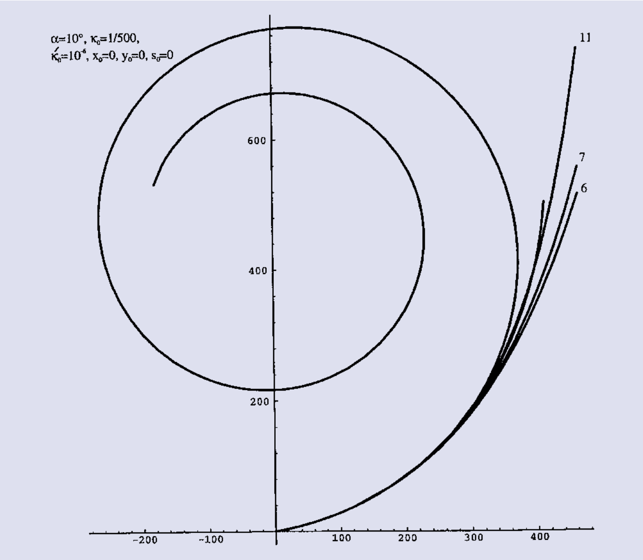

Figure 1.7: Local representation of the circle by (1.23), κ_0 = 1/500, κ_0' = 10^{-6}, α_0 = 10°, x_0 = y_0 = 0, s_0 = 0, various degrees of approximation.

High-speed-railways/bullet trains/Transrapids/ need "*extreme reliable control systems*" (*extasy*) to avoid catastrophic events. Here we consequently focus on two sensitive problems related to *extasy*: (*i*) a local high resolution representation of the track design (clothoid, circle, straight line) in UTM *map matching coordinates* is given. (*ii*) The *Mixed Model* (*universal Kriging*) is used to discriminate measurement errors from track displacements. *Part I* reviews the local representation of the clothoid (special case: circle straight line) which is needed in the expert case of *extasy*.

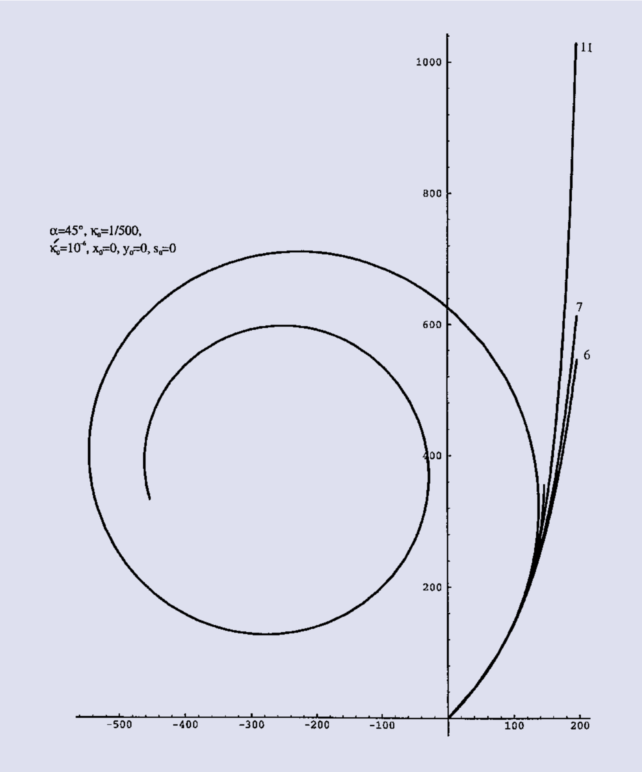

Figure 1.8: Local representation of the circle by (1.23), κ_0 = 1/500, κ_0' = 10^{-6}, α_0 = 45°, x_0 = y_0 = 0, s_0 = 0, various degrees of approximation.

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Hochgeschwindigkeitstrassen, sog. Stoßwellenzüge/Transrapid/ benötigen extrem zuverlässige Überwachungssysteme („*extasy*"), um katastrophale Unglücksfälle zu vermeiden. Auf dem Weg zu „extasy" behandeln wir zwei Sensibilitätsprobleme: (*i*) wir entwickeln eine hochauflösende Darstellung des Trassenentwurfes (Klothoide, Kreis, Gerade) in UTM / Gauß-Krüger „*Map Matching*" Koordinaten. (*ii*) Das zugeordnete *Gemischte Modell* (*universelles „Kriging"*) wird vorgestellt, insbesondere um Messfehler von einer Trassenverschiebung zu trennen. Der erste Teil konzentriert sich auf die lokale, hochauflösende Darstellung der *Klothoide* (Spezialfall: Kreis, Gerade), Grundlage für den Aufbau einer trassenorientierten Wissensbasis.