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Biased Estimation in Gauss-Markov Model with Constraints

Abstract

The biased estimation problem in Gauss-Markov model with constraints is considered when some multicollinearities exist among the columns of the design matrix. A class of restricted biased estimators are defined by grafting the (unrestricted) biased estimation philosophy into the restricted least squares (RLS) estimator, and some important properties are established. As special cases of our class, five known (unrestricted) biased estimators are extended to Gauss-Markov model with constraints. At last, a numerical example is used to illustrate the higher accuracy of the proposed restricted biased estimators over the RLS estimators.

1 Introduction

A serious problem that often occurs in geodetic data processing is the presence of multicollinearity among the columns of the design matrix, causing highly unstable least squares (LS) estimators of the unknown parameters. As alternatives to the LS estimator, many biased estimators, such as ordinary ridge estimator, principal components estimator, combining ridge and principal components estimator, combined principal components estimator, single-parametric principal components estimator and latent root estimator etc. (cf. GUI, DUAN and ZHU 1999; GUI and LIU 2000) have been proposed and shown to be a quite satisfactory solution to the multicollinearity.

The biased estimation problem in Gauss-Markov model with constraints is considered when some multicollinearities exist among the columns of the design matrix in this paper. We propose a class of restricted biased estimators by grafting the (unrestricted) biased estimation philosophy into the restricted least squares (RLS) estimator, and establish some important properties. Many well-known (unrestricted) biased estimator, e.g. ordinary ridge estimator, principal components estimator, combining ridge and principal components estimator, combined principal components estimator, single-parametric principal components estimator etc. are extended to Gauss-Markov model with constraints. A numerical example is presented to illustrate that these restricted biased estimators are better than the RLS estimator when some multicollinearities exist.

2 Restricted generalized shrunken least squares estimation

We consider the Gauss-Markov model with constraints

$$\begin{cases} L = AX + \Delta \\ CX = W \\ \mathrm{E}(\Delta) = 0, \mathrm{Cov}(\Delta) = \sigma_0^2 P^{-1} \end{cases} \tag{1}$$

where L is an $n \times 1$ vector of observations with an $n \times n$ positive definite weight matrix P, A is an $n \times t$ design matrix with rank $(A) = t$, X is a $t \times 1$ vector of unknown parameters, Δ is an observation error vector, C is an $s \times t$ matrix of known coefficients with rank $(C) = s$ and W is an $s \times 1$ vector of known constants.

It is well known that the restricted least squares (RLS) estimator of X denoted by $\hat{X}_{RLS}$, which is obtained by minimizing $\Delta^T P \Delta$ subjected to the constraints $CX = W$, can be written as

$$\hat{X}_{RLS} = \hat{X}_{LS} + N^{-1} C^T (CN^{-1} C^T)^{-1} (W - C\hat{X}_{LS}) \tag{2}$$

where $N = A^T PA$ and $\hat{X}_{LS} = N^{-1} A^T PL$ is the unrestricted LS estimator for X (cf. KOCH 1987; RAO and TOUTENBURG 1995).

It is easily to prove that when some multicollinearities exist among the columns of the design matrix A, the RLS estimator, same as the unrestricted LS estimator, is no longer a good estimator. In response to the perceived deficiencies with the RLS estimator, we propose a new estimator by grafting the generalized shrunken least squares (GSLS) estimation technique (cf. GUI and LIU 2000) philosophy into the RLS estimator, which may be designated the restricted generalized shrunken least squares (RGSLS) estimator and be denoted by $\hat{X}_{RGSLS}(D)$:

$$\hat{X}_{RGSLS}(D) = QDQ^T \hat{X}_{RLS} \tag{3}$$

where $D = \mathrm{diag}(d_1, \ldots, d_t)$ is called a generalized shrinking parameter matrix, Q is an orthogonal matrix such that $Q^T NQ = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_t)$. We assume, without any loss of generality, that $\lambda_1 \geq \cdots \geq \lambda_t > 0$.

Obviously, the RLS estimator refers to the case where $D = I_t$. It is easily seen from (3) that $\hat{X}_{RGSLS}(D)$ is always a restricted biased estimator of X unless $D = I_t$. Further, according to concrete problems in geodetic data processing, we can generate many useful restricted biased estimators by appropriate choices of the generalized shrinking parameter matrix D. Several important estimators are given next.

1. When $D = \Lambda\,(\Lambda + kI_t)^{-1}$, we get

$$\hat{X}_{ROR}(k) = \hat{X}_{RGSLS}\,(\Lambda\,(\Lambda + kI_t)^{-1}) = (I_t + kN^{-1})^{-1}\,\hat{X}_{RLS} \quad (4)$$

which is called a restricted ordinary ridge (ROR) estimator, where $k > 0$ is called a ridge parameter.

2. When

$$D = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

we get

$$\hat{X}_{RPC}(r) = \hat{X}_{RGSLS}\,(\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}) = Q_r Q_r^T\,\hat{X}_{RLS} \quad (5)$$

which is called a restricted principal components (RPC) estimator, where

$$Q_r = Q\begin{pmatrix} I_r \\ 0 \end{pmatrix}, r = 1, \ldots, t.$$

3. When

$$D = \begin{pmatrix} \Lambda_r\,(\Lambda_r + kI_r)^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

we get

$$\hat{X}_{RCRPC}(k, r) = \hat{X}_{RGSLS}\,(\begin{pmatrix} \Lambda_r\,(\Lambda_r + kI_r)^{-1} & 0 \\ 0 & 0 \end{pmatrix})$$

$$= Q_r\,\Lambda_r\,(\Lambda_r + kI_r)^{-1}\,Q_r^T\,\hat{X}_{RLS} \quad (6)$$

which is called a restricted combining ridge and principal components (RCRPC) estimator, where $\Lambda_r = \text{diag}\,(\lambda_1, \ldots, \lambda_r)$ and $k > 0$.

4. When

$$D = \begin{pmatrix} \Lambda_m^{-1} & 0 \\ 0 & \Lambda_{t-m} \end{pmatrix},$$

we get

$$\hat{X}_{RCPC} = \hat{X}_{RGSLS}\,(\begin{pmatrix} \Lambda_m^{-1} & 0 \\ 0 & \Lambda_{t-m} \end{pmatrix}) = Q\begin{pmatrix} \Lambda_m^{-1} & 0 \\ 0 & \Lambda_{t-m} \end{pmatrix}Q^T\,\hat{X}_{RLS} \quad (7)$$

which is called a restricted combining principal components (RCPC) estimator, where $\Lambda_m = \text{diag}\,(\lambda_1, \ldots, \lambda_m)$, $\Lambda_{t-m} = \text{diag} = (\lambda_{m+1}, \cdots, \lambda_t)$, and suppose there exists $1 \le m \le t$ such that $\lambda_m > 1 \ge \lambda_{m+1}$.

5. When

$$D = \begin{pmatrix} I_m - (1-\theta)\,\Lambda_m^{-1} & 0 \\ 0 & \theta\Lambda_{t-m} \end{pmatrix},$$

we get

$$\hat{X}_{RSPPC}(\theta) = \hat{X}_{RGSLS}\,(\begin{pmatrix} I_m - (1-\theta)\,\Lambda_m^{-1} & 0 \\ 0 & \theta\Lambda_{t-m} \end{pmatrix})$$

$$= Q\begin{pmatrix} I_m - (1-\theta)\,\Lambda_m^{-1} & 0 \\ 0 & \theta\Lambda_{t-m} \end{pmatrix}Q^T\,\hat{X}_{RLS} \quad (8)$$

which is called a restricted single-parametric principal components (RSPPC) estimator, where $\theta \in (0,1)$ is called a stable parameter.

Obviously, each estimator given here is the application of the recently developed unrestricted biased estimation theory in Gauss-Markov model to one with constraints, and with which methods of estimating parameters in Gauss-Markov model with constraints will be greatly improved.

We can prove the following properties of the RGSLS estimator by using the same method as Gui and Liu (2000) did.

Theorem 1. If

$$X^T\,TX < \sigma_0^2$$

then

$$\text{MSE}\,(\hat{X}_{RGSLS}\,(D)) < \text{MSE}\,(\hat{X}_{RLS})$$

that is, the RGSLS estimator $\hat{X}_{RGSLS}\,(D)$ is superior to the RLS estimator in the sense of the reduced MSE, where

$$T = Q\,(I_t - D)\,(Q^T MQ - DQ^T MQD)^{-1}\,(I_t - D)\,Q^T$$

$$M = N^{-1} - N^{-1}\,C^T\,(CN^{-1}\,C^T)^{-1}\,CN^{-1}$$

Theorem 2. The RGSLS estimator $\hat{X}_{RGSLS}\,(D)$ is admissible for X in the class of all linear estimators under quadratic loss if and only if

$$0 \le d_i \le 1,\; i = 1, \ldots, t$$

that is, there does not exist a linear estimator which is uniformly better than the RGSLS estimator in terms of the MSE.

Numerical example

For illustrative purpose, we consider a geodetic network as shown in Fig. 1. The angles $l_1, \ldots, l_{12}$ are measured and have the following values:

$l_1 = 32°49'18.05''$	$l_5 = 75°52'53.00''$	$l_9 = 31°25'15.15''$
$l_2 = 22°04'22.82''$	$l_6 = 20°00'15.41''$	$l_{10} = 112°13'30.06''$
$l_3 = 125°06'18.91''$	$l_7 = 130°50'20.87''$	$l_{11} = 39°05'39.36''$
$l_4 = 84°06'50.16''$	$l_8 = 17°44'21.52''$	$l_{12} = 28°40'46.51''$

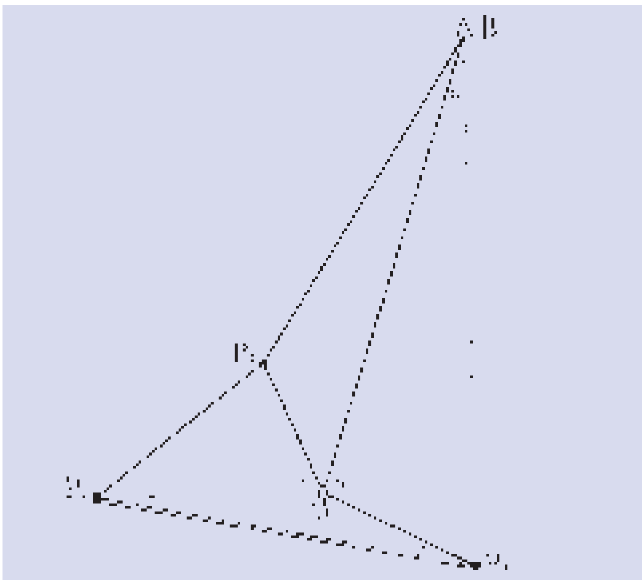

Fig. 1: A simulated geodetic network

All observations are uncorrelated with equal weight and the weight of every angle is one. The standard deviation of each angle observation is assumed to be $m_\alpha = \pm 1.50''$. The coordinates of two known control points A and B are listed as follows:

$$X_A = 6504899.84\ (m) \quad Y_A = -49869.19\ (m)$$

$$X_B = 6508781.94\ (m) \quad Y_B = -48900.56\ (m)$$

The distance between P_1 and P_3 is known and is equal to 22979.56 (m).

We can easily get A, L, C and W as

$$A = \begin{pmatrix} 0.49 & 0.56 & -0.12 & 0.50 & -0.36 & -1.06 \\ -0.12 & 0.50 & -0.33 & 0.13 & -0.46 & -0.63 \\ -0.36 & -1.06 & 0.46 & -0.63 & -0.10 & 1.69 \\ 0.02 & -1.15 & 0.00 & 0.00 & 0.36 & 1.06 \\ 0.36 & 1.06 & 0.00 & 0.00 & -0.69 & -0.85 \\ -0.39 & 0.09 & 0.00 & 0.00 & 0.32 & -0.21 \\ -0.93 & -0.31 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.39 & -0.09 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.55 & 0.40 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.42 & 0.90 & 0.12 & -0.50 & 0.00 & 0.00 \\ -0.55 & -0.40 & 0.09 & 0.34 & 0.00 & 0.00 \\ 0.12 & -0.50 & -0.21 & 0.16 & 0.00 & 0.00 \end{pmatrix}$$

$$L = (-0.95, 0.82, -0.09, 1.16, -0.99, -1.59, -0.13, -0.47, -1.85, -0.94, -1.80, -1.33)^T$$

$$C = (0.00, 0.00, 1.27, -15.07, -1.27, 15.07)$$

$$W = -0.0005$$

Some multicollinearities between the columns of the design matrix A are serious, which is reflected in the condition number of the normal equation matrix $A^T PA$, $k = 144.6$. Tab. 1 lists the estimates of the increments δx and δy to the approximate coordinates of three unknown points P_1, P_2 and P_3 by using the RLS estimation and RGSLS estimation, respectively, where the approximate coordinates assigned to points P_1, P_2 and P_3 to begin with. It is obvious to see that the RGSLS estimator is better than the RLS one when some multicollinearities exist.

Tab.1. Coordinate corrections of the three unknown points

	δx_1	δy_1	δx_2	δy_2	δx_3	δy_3
RLSE	−0.21	0.20	−1.76	0.56	0.43	0.75
RORE ($k = 1.58$)	−0.15	0.12	−0.20	0.21	0.41	0.29
RPCE ($r = 1$)	−0.05	−0.11	0.02	−0.03	0.02	0.12
RCRPCE ($r = 2, k = 1$	0.01	0.04	0.03	−0.04	0.01	0.02
RCPCE ($m = 3$)	0.09	0.04	−0.43	0.54	0.68	0.22
RSPPCE ($\theta = 0.01$)	−0.26	0.15	−0.02	−0.01	0.21	0.21

Acknowledgements

The project is sponsored by Natural Science Foundation of China (49825107) and it is partly supported by Natural Science Foundation of Henan Province, China (004051300).

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Analyse der Entwicklung der russischen Kartographie

Der 1938 erschossene sowjetische Wirtschaftswissenschaftler N. D. Kondrat'ev begründete eine Planungstheorie, nach der sich die Weltwirtschaft quasiperiodisch mit Höhen und Tiefen alle 50 Jahre entwickelt. Auf dieser Grundlage wurde die Evolution der russischen Kartographie seit 1701 anhand der staatlichen Organisationsstruktur, der Technologien für die astronomischen, geodätischen und kartographischen Arbeiten, der Technik und der Geräte sowie der Ausbildung der Fachkräfte mit Hilfe von 200 Daten aus bekannten historischen Quellen in Gruppen von 10 Jahren dargestellt (Abb. 1). Die durchgehende Linie gibt die Dynamik der Ökonomie nach Kondrat'ev an, ergänzt durch Untersuchungen von Frank und Mandel.

Der Vergleich der Ergebnisse mit den Kondrat'evschen Zyklen zeigt, dass die maximale Zahl wesentlicher Ereignisse an den Spitzen der ökonomischen Entwicklung liegt. Die Interpolation der Kondrat'evschen Daten deutet auf Krisiserscheinungen Ende des 20. Jahrhunderts hin.

Aus: Evoljucija otečestvennoj kartografii v svete teorii dlinnych voln Kondrat'eva. Von Tarelkin, E. P. – Geodez. i Kartogr. Moskva (2000) 5, S. 38–39

Deumlich

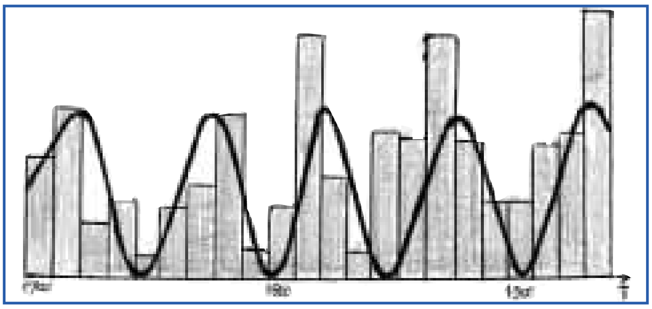

Abb. 1: Evolutionszyklen der Ökonomie und Kartographie