Refraction in Bilateral Trigonometric Leveling – Definition of Corrections

Die Refraktion beim gegenseitigen trigonometrischen Nivellement – Definition von Korrektionen

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This article is dealing with problems related to the application of the similarity theory in order to reduce the influence of refraction in trigonometric leveling. After a brief introduction of the basics the determination of corrections due to refraction in two sided trigonometric leveling is explained, a method which is based on the simulation of the parameters $\rho_\Sigma = \rho_1 + \rho_2$ and $\Delta\rho = \rho_1 - \rho_2$. These parameters are obtained by joint geodetic and metrological measurements and are caused by the impact of the refraction on ρ_1 and ρ_2.

Keywords: Refraction, leveling, theory of similarity, modeling, corrections, verification, experiment

Dieser Beitrag behandelt Fragestellungen, die mit der Anwendung der Ähnlichkeitstheorie zur Reduktion des Refraktionseinflusses verbunden sind. Neben einer zusammenfassenden Darstellung der Grundlagen wird insbesondere die Bestimmung von Refraktionskorrektionen beim gegenseitigen trigonometrischen Nivellement erläutert, einer Methode, die auf der Simulation der Parameter $\rho_\Sigma = \rho_1 + \rho_2$ und $\Delta\rho = \rho_1 - \rho_2$. basiert. Diese Parameter werden durch gemeinsame geodätische und meteorologische Messungen erhalten, hervorgerufen durch den Refraktionseinfluss auf ρ_1 und ρ_2.

Schlüsselwörter: Refraktion, trigonometrisches Nivellement, Ähnlichkeitstheorie, Modellbildung. Korrektionen, experimentelle Verifikation

1 INTRODUCTION

The determination of corrections due to the influence of refraction in one-sided leveling is based on the establishment of a causal relationship between the integral gradient of the air temperature $\overline{\gamma}_z = \dfrac{\partial T}{\partial z}$ in the propagation path of the optical beam with the height difference $\rho_0 = h - h_0$ between the visible target image and its actual position /Možuchin 1992/. The gradient sign is considered positive when the air temperature decreases with altitude. From simple geometric considerations (see *Fig. 1*) it follows that

$$\frac{\rho_0}{S} = \frac{S}{R_0}, \tag{1}$$

the ratio of the height difference ρ_0, obtained by geodetic measurements, to the arc S (the distance between the observation and

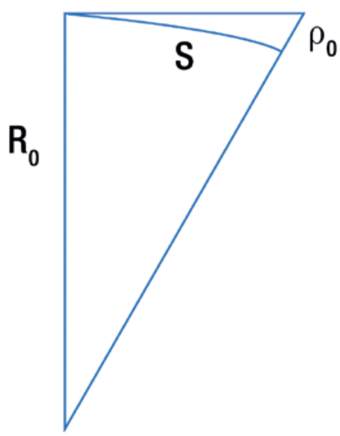

Fig. 1 | Scheme of the influence of refraction in trigonometric leveling sighting points) is equal to the ratio of the same arc of the optical ray trajectory to its radius R_0. The reciprocal of the latter is the integral gradient of the refractive index along the path of the ray /Možžuchin 1995/:

$$\overline{\gamma}_N = R_0^{-1} = -\frac{\mathrm{d}n}{\mathrm{d}z} = -\left(\frac{\mathrm{d}N}{\mathrm{d}z}\right)\cdot 10^6. \tag{2}$$

The gradient of the refractive index is determined by the gradients of atmospheric pressure and air temperature. If the pressure always falls regularly with altitude, then the temperature can both decrease and increase, thereby determining the problem of taking refraction into account in trigonometric leveling. Given the known relationship between the refractive index gradient and the meteorological parameters of the atmosphere /Mozzhukhin 2008/, and after substituting into Eq. (*1*), we obtain:

$$\begin{aligned}\rho &= 79\,p\,T^{-2}\,S^2(0.0342-\overline{\gamma}_z)\cdot 10^{-6}\\ &\approx S^2(0.0342-\overline{\gamma}_z)\cdot 10^{-6},\end{aligned} \tag{3}$$

where p and T are the atmospheric pressure (hPa) and air temperature (K), respectively. In the "standard" atmosphere (p = 1013 hPa, T = 288 K), the factor in front of the brackets is equal to 0.965, which allows us to use the approximate formula. The presence of the measured value ρ_0 makes it possible to determine the integral temperature gradient from the formula:

$$\begin{aligned}\overline{\gamma}_z &= \left(0.0342-\frac{\rho_0\;T^2}{79\,P\,S^2}\right)\cdot 10^6\\ &\approx \left(0{,}0342-\rho_0\,S^{-2}\right)\cdot 10^6.\end{aligned} \tag{4}$$

In contrast to the integral gradient, its analogue is the "local" temperature gradient /Možžuchin 1995/:

$$\gamma_z = a\cdot\gamma_1^{0.2}\cdot z_{\mathrm{m}}^{-1}. \tag{5}$$

Using the values γ_1 and z_{m} easily accessible for direct measurements (temperatures to determine the temperature gradient γ_1 at height z = 1 m and heights to compute the average height z_{m} of the sight beam above the earth's surface) and after substituting Eq. (*5*) into Eq. (3), it becomes possible to determine corrections based on meteorological measurements. The method for determining the gradients γ_1 and heights z_{m} is considered in /Mozzhukhin 2008/. The transition factor (the coefficient of such a transformation) is found by comparing simultaneously measured temperature gradients $\overline{\gamma}_z$ and γ_z:

$$a = \frac{\overline{\gamma}_z}{\gamma_z} = \frac{\overline{\gamma}_z\cdot z_{\mathrm{m}}}{\gamma_1^{0.2}}. \tag{6}$$

The theoretical basis of the method for determining corrections is the theory of similarity, dimensional analysis and modeling /Pawlowski 2013/, which is widely used to solve a wide range of practical problems associated with the motion of a liquid or gas. This theory is based on three theorems. One of them (π-theorem) testifies that any equation reflecting the physical process can be represented in a dimensionless form. This reduces the number of arguments under the sign of the function, and, equally important, there is an opportunity to transfer the results of the experiment obtained on one object (model) to other objects (the original), regardless of the place and time of the experiment. The model is always an incomplete copy of the original, reflecting its most significant aspects. Two other theorems formulate conditions – necessary and sufficient – for observing the similarity of physical phenomena.

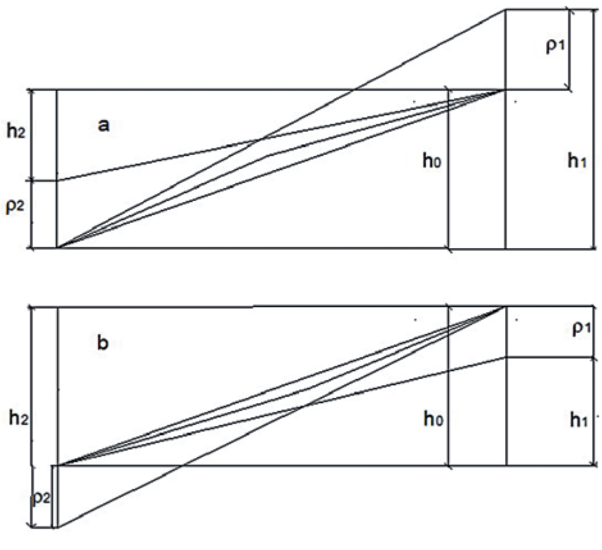

Fig. 2 | Scheme of the influence of refraction in bilateral trigonometric leveling with the arrangement of the sight beams above the auto convection layer (a) and inside it (b)

2 INFLUENCE OF REFRACTION IN BILATERAL TRIGONOMETRIC LEVELING

The result of measuring the excess by two-sided trigonometric leveling $h_{\mathrm{m}} = 0.5\cdot(h_1 + h_2)$ contains the known (measured) height difference $(h_1 - h_2)$ which is equal to the sum $\rho_\Sigma = \rho_1 + \rho_2$ or the difference $\Delta\rho = \rho_1 - \rho_2$ caused by the refraction of ρ_1 and ρ_2. h_1 and h_2 are the observations made in the direction of the positive and negative slopes. The sum of the systematic errors ρ_Σ arises in case of non-realities of heights $h_1 > h_0 > h_2$ and their difference $\Delta\rho$ under the conditions of another type of $h_1 > h_2 > h_0$, where h_0 is the real height, free from the influence of refraction. In experimental studies of refraction, the latter is found by precise geometric leveling.

According to the inequality of the first type, we get: $\rho_1 = h_1 - h_0$ and $\rho_2 = h_0 - h_2$ for the second: $\rho_1 = h_1 - h_0$ and $\rho_2 = h_2 - h_0$. Hence, the difference in heights $(h_1 - h_2)$ corresponds in the first case to the sum of the parameters ρ_Σ, and in the second case to their difference $\Delta\rho$. Then the required quantities for the introduction of corrections are $\Delta\rho$ and ρ_Σ, respectively.

A number of physically homogeneous quantities found by means of geodetic measurements, ρ_1'', ρ_2'' as well as their derivatives $\Delta\rho''$, ρ_Σ'' correspond to analogous quantities ρ_1', ρ_2', $\Delta\rho'$, ρ_Σ' obtained on the basis of meteorological measurements.

The degree of similarity of the analogue (with one stroke) and the original is determined by the coefficients $k_1 = \rho_1''/\rho_1'$, $k_2 = \rho_2''/\rho_2'$, $k_\Delta = \Delta\rho''/\Delta\rho'$, $k_\Sigma = \rho_\Sigma''/\rho_\Sigma'$. The parameters $\Delta\rho$ and ρ_Σ combine the transition factor $C = \Delta\rho/\rho_\Sigma$, and the quantities C' and C'' are the coefficients $k_C = C''/C'$. Dividing the right and left parts of the derived relations we get:

$$\frac{C'\Delta\rho''\rho_\Sigma'}{C''\Delta\rho'\rho_\Sigma''}) = 1 \text{ and } \frac{k_\Delta}{k_C\,k_\Sigma} = 1,$$

which serves as a condition for maintaining the similarity of the model and the original. In this case, the choice of coefficients must satisfy the condition $k_\Delta = k_\Sigma k_C$.

The sequence of numbers ρ_1, ρ_2 can be considered as members of an arithmetic progression with the difference $\Delta\rho$ and also a geometric progression with the denominator $q = \rho_2/\rho_1$. If we use the denominator $q = 0.618$ (golden section), then each successive number of terms of the series ρ_Σ, ρ_1, ρ_2, $\Delta\rho$ is obtained from the previous one by multiplying with 0.618.

3 EXPERIMENTAL STUDIES

Tab. 1 presents the results of bilateral leveling on four sides of a closed polygon with a perimeter of 27 km.

Columns 3 and 4 depict the height differences measured for ascent and descent, and in columns 5 and 6 their differences and mean values are given. The signs of the latter are indicated in the course of motion relative to the starting point. Three of them, measured in the direction of a positive slope, are indicated with a plus sign. In accordance with this, a discrepancy is obtained in line 7. The latter is equal to the half-difference of the residuals in columns 3 and 4 (line 7), and their algebraic difference is the sum of the numbers in column 5. Exceeds in column 7 are measured by precise geometric leveling, which allows to obtain the values in column 8 caused by refraction (systematic errors). Their sum (taking into account the sign in column 6) is equal to the discrepancy. As it can be seen, the errors depend on the difference in heights between the determined and the starting points of the move. They are compensated among themselves and have little effect on the magnitude of the discrepancy.

It should be noted that, by analogy with discrepancies of a closed stroke, the discrepancies d in the courses of high-precision leveling are not enough to estimate the accuracy of the measured heights.

The missing parameters ($\Delta\rho''$ and Δ''_Σ) in *Tab. 1* are presented in *Tab. 2* (columns 7 and 8) in the form of its analog. A comparison of columns 7 and 9 allows us to conclude that in column 5 of *Tab. 1* the values of the parameter ρ''_Σ are indicated. The exception is the data of line 2, obtained at the height of the auto-convection layer ($z_\mathrm{m} \sim z^0$) where the calculation of the ρ parameters becomes uncertain. Calculation of the height z^0 by the formula /Možžuchin 1995/:

$$z^0 = \frac{(\gamma'_1)^{0,2} + (\gamma''_1)^{0,2}}{0.0684} = \frac{\gamma_\mathrm{m}^{0.2}}{0.0342} \tag{7}$$

leads to a value of 12.5 m. In this case, $z_\mathrm{m} = 12.0$ m.

The data in the columns 2 and 3 allows us to determine the temperature gradients γ_z by Eq. (*5*) with a factor $a = 1.0$, and then the gradients of the refractive index γ_N. After multiplying the latter by S^2 the values ρ'_1 and ρ'_2 in the columns 5 and 6 are obtained. So, for example, according to the data of line 1 in the first stage, we have 0.0254; 0.0221 K/m. On the second –0.0088; 0.0121 m^{-1}. After multiplying by the square of the distance (5.367 km) we obtain the values in columns 5 and 6 of *Tab. 2*.

The coefficients of such a transformation $\rho''_\Sigma / \rho'_\Sigma$ in *Tab. 1* are equal to: 0.90; –7.90; 2.02; 1.35. The lack of similarity between the known and meteorological parameters calculated in line 2 indicates a weak effect of refraction. In this connection, there is no need to introduce an amendment. In the remaining three cases, the desired differences can be determined from one of the two relations:

$$\Delta\rho'_1 = \rho'_1 - \left(-\rho'_1\right) = 2\rho'_1 - \rho''_\Sigma \text{ or}$$

$$\Delta\rho'_2 = \left(\rho''_\Sigma - \rho'_2\right) - \rho'_2 = \rho''_\Sigma - 2\rho'_2, \tag{8}$$

which allows to reject one of the obtained results. So, according to the data given in *Tab. 2*, we have: –0.033 and –0.157 m; –0.081 and 1.675 m; 0.230 and 1.022 m. Finally, we get: 0.017; 0.042; 0.115 m. The corrections $\delta = 0.5\Delta\rho$ to the measurement results h_m are introduced with the plus sign, since two-sided measurements were performed under the constraint $\rho'_1 < \rho'_2$. Corrected results are indicated in column 9 of *Tab. 1*. The magnitude of the systematic error due to refraction in the open stroke was 0.230 m. After the introduction of corrections, it decreased to –0.040 m. Similar measurements in a closed loop of seven points are presented in *Tab. 3* and *Tab. 4*.

The comparison of the parameters ρ'_Σ and $\Delta\rho'$ in *Tab. 4* with the known height difference ($h_1 - h_2$) indicates that in two cases (lines 1 and 7) there are differences $\Delta\rho''$ and in the remaining, except for line 4, the sum of ρ''_Σ measured values ρ''_1, ρ''_2. From *Tab. 3* (column 8) it follows that systematic errors in the presence of differences are an order of magnitude greater than those where sums of numbers are present.

The similarity coefficients $k_\Delta = \Delta\rho''/\Delta\rho'$ in the lines 1 and 7 of *Tab. 4* are 0.43 and 1.56, respectively. Multiplying the last by the terms of the corresponding row of numbers in columns 5–8, respectively, we obtain: 0.470; 0.310; 0.780; 0.160 m and 0.554; 0.379; 0.933; 0.175 m. Thus, we have two series of arithmetic progression numbers with differences $\Delta\rho''$, obtained from geodetic measurements.

Any member of an arithmetic progression can be found by the formula:

$$\rho_n = \rho_1 + (n-1)\Delta\rho''.$$

So, for example, for $\rho_1 = 0.470$ m and $n = 6$ we get $\rho_n = 1.270$ m. Assuming $\rho_1 = 0.310$ m and $n = 7$, we also obtain $\rho_n = 1.270$ m. Turning to the initial data $\rho'_1 = 1.094$ m, $\rho'_2 = 0.722$ m in line 1 of *Tab. 4*, by analogy we find 1.254 m and 1.203 m. The average of them is 1.228 m. The correction to the result of bilateral leveling h_m is equal to $\delta = -0.614$ m. In the same way, using the data in line 7, we get 1.230 m and 1.293 m, and the average of them is 1.262 m. Hence, $\delta = -0.631$ m.

The data of lines 3 and 6 indicate the presence of gradients γ_1 that are identical in magnitude and sign at the line ends, and, consequently, the weakening of the effect of refraction on the results of leveling, taking into account that the parameters $\Delta\rho$ and ρ_Σ are related. If the quantity $\rho''_\Sigma = 0{,}248$ m in line 3 is much less than the corresponding value of 1.443 m in line 6, then the same effect should be expected in relation to $\Delta\rho$. It follows that the correction can be neglected in the first case and tried to introduce it in the second. Expanding $\rho''_\Sigma = 1.443$ m in a series of geometric progression with denominator $q = 0.618$, we obtain $\Delta\rho' = 0.340$ m. Since the gradients γ_1 are equal that is the basis to use half of this value, which corresponds to the correction $\delta = -0.085$ m.

After the expansion of the known parameters 0.313 and 0.345 m into rows of 4 and 5, we obtain the differences $\Delta\rho'$, equal to 0.074 m and 0.081 m, corresponding to the corrections –0.037 and –0.040 m. Calculation of these differences according to formulas (Eq. (*8*)) leads to two digits (0.055, 0.423 m) in the first case and 0.081; 0.097 m in the second one. Taking the first of the two digits for the final result, let us pay attention to a satisfactory coincidence with the preliminary calculation in one case and the complete coincidence in the other one.

The data in line 2 indicate that bilateral measurements were made within the boundaries of the auto-convection layer (z_0 = 16.7 m). Calculation by Eq. (*8*) allows us to find the correction δ = –0.042 m. The total systematic error in the open loop (column 8 of *Tab. 3*) was 1.870 m, after the introduction of corrections 0.041 m.

No.	S	h_1	h_2	$h_1 - h_2$	h_m	h_0	$h_m - h_0$	$h_{cor.} - h_0$
1	2	3	4	5	6	7	8	9
1	5367 m	13.540 m	13.001 m	0.539 m	13.270 m	13.296 m	–0.026 m	–0.009 m
2	3660 m	11.210 m	10.929 m	0.281 m	11.070 m	11.066 m	0.004 m	0.004 m
3	9515 m	14.982 m	12.655 m	2.327 m	13.818 m	13.903 m	–0.085 m	–0.043 m
4	8433 m	38.909 m	37.375 m	1.534 m	–38.142 m	–38.265 m	–0.123 m	–0.008 m
5	26980 m	39.732 m	36.585 m	3.147 m	38.158 m	38.265 m	–0.107 m	–0.048 m
6	–	–38.909 m	–37.375 m	–1.534 m	–38.142 m	–38.265 m	0.123 m	–0.008 m
7	–	0.823 m	–0.790 m	1.613 m	0.016 m	0.000 m	0.016 m	–0.056 m

Tab. 1 | Results of bilateral leveling of the closed stroke

No.	γ'_1	γ''_1	z_m	ρ'_1	ρ'_2	ρ'_Σ	$\Delta\rho'$	$h_1 - h_2$
1	2	3	4	5	6	7	8	9
1	0.02 K/m	0.01 K/m	18,0 m	0.253 m	0.348 m	0.601 m	–0.095 m	0.539 m
2	0.01 K/m	0.02 K/m	12,0 m	0.013 m	–0.052 m	–0,039 m	0.065 m	0.281 m
3	0.02 K/m	0.11 K/m	21,0 m	1.123 m	0.326 m	1.149 m	0.797 m	2.327 m
4	0.02 K/m	0.11 K/m	21,0 m	0.882 m	0.256 m	1.138 m	0.626 m	1.534 m

Tab. 2 | Results of determining the parameters ρ'_1 and ρ'_2 using meteorological measurements (model)

No.	S	h_1	h_2	$h_2 - h_1$	h_m	h_0	$h_m - h_0$	$h_{cor.} - h_0$
1	2	3	4	5	6	7	8	9
1	8718 m	10.392 m	10.232 m	0.160 m	10.312 m	9.706 m	0.606 m	–0.008 m
2	2952 m	4.673 m	4.761 m	–0.088 m	4.717 m	4.631 m	0.086 m	0.044 m
3	4134 m	9.443 m	9.195 m	0.248 m	9.319 m	9.324 m	–0.005 m	–0.005 m
4	6070 m	62.977 m	62.664 m	0.313 m	62.820 m	62.771 m	0.049 m	0.012 m
5	5868 m	16.789 m	16.444 m	0.345 m	16.616 m	16.577 m	0.039 m	–0.001 m
6	9669 m	14.239 m	12.796 m	1.443 m	13.518 m	13.442 m	0.076 m	–0.009 m
7	5312 m	61.884 m	61.709 m	0.175 m	61.796 m	61.157 m	0.639 m	0.008 m
8	42,7 km	90.239 m	88.461 m	1.778 m	89.350 m	88.614 m	0.736 m	0.038 m
9	–	–90.158 m	–89.340 m	–0.818 m	–89.748 m	–88.614 m	–1.134 m	–0.003 m
10	–	0.081 m	–0.879 m	–0.960 m	–0.398 m	0.000 m	–0.398 m	0.035 m

Tab. 3 | Results of bilateral leveling of a closed test site

No.	γ'_1	γ''_1	z_m	ρ'_1	ρ'_2	ρ'_Σ	$\Delta\rho'$	$h_1 - h_2$
1	2	3	4	5	6	7	8	9
1	0.03 K/m	0.09 K/m	25 m	1.094 m	0.722 m	1.816 m	0.372 m	0.160 m
2	0.04 K/m	0.09 K/m	14 m	–0.029 m	–0.086 m	–0.115 m	0.057 m	–0.088 m
3	0.04 K/m	0.04 K/m	18 m	0.085 m	0.085 m	0.170 m	0.000 m	0.248 m
4	0.04 K/m	0.11 K/m	18 m	0.184 m	–0.055 m	0.129 m	0.239 m	0.313 m
5	0.07 K/m	0.11 K/m	21 m	0.213 m	0.124 m	0.337 m	0.089 m	0.345 m
6	0.07 K/m	0.07	25 m	1.000 m	1.000 m	2.000 m	0.000 m	1.443 m
7	0.03 K/m	0.07	23 m	0.355 m	0.243 m	0.598 m	0.112 m	0.175 m

Tab. 4 | Results of determining the parameters ρ'_1 and ρ'_2 on the model (by meteorological measurements)

4 CONCLUSION

Thus, by eliminating the influence of refraction, the accuracy of two-sided trigonometric leveling is improved comparable to the results of precise geometric leveling, without additional costs and with very simple means.

In science, everything can be measured and evaluated, and a scientific result is the one which has multiple experimental confirmation. A negative result avoids the repetition of errors. Conclusions and recommendations based on empirical representations are usually useless or harmful. The method of similarity, dimensional analysis and modeling, along with the physical picture of the phenomenon – the theoretical basis of the method of determining corrections due to the influence of refraction in leveling – is important for the professional training of geodetic specialists.

The author expresses deep gratitude to Prof. Dr.-Ing. Bertold Witte for many years of fruitful cooperation.

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NACHWORT

Herr Prof. Dr. Oleg A. Mozzhukhin konnte die sprachlichen Veränderungen und kleineren Korrekturen, welche die Schriftleitung und ich zum leichteren Verständnis seines Beitrags vorgenommen haben, nicht mehr autorisieren, da er Anfang dieses Jahres seinem schweren Leiden erlag.

Er zählte zu den renommierten russischen Kollegen, die sich theoretisch und praktisch mit der Erfassung und Berücksichtigung des Refraktionseinflusses auf die Ergebnisse des geometrischen und trigonometrischen Nivellements auseinandergesetzt haben. Seit 2001 hat unser hoch geschätzter Kollege in den avn mehrere Beiträge zu dieser Problematik veröffentlicht.

Seine wissenschaftlichen Leistungen bleiben unvergessen.

Bertold Witte