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Analysis of the accuracy of estimation results obtained by the method of least absolute deviations

This paper presents an analysis of the accuracy of estimation results obtained by the method of least absolute deviations. Theoretical considerations have been supplemented with an example of use on a simulated levelling network.

1 Introduction

In his paper (VUKTOTIĆ, 1982) VUCTOTIĆ proposed a method of the least absolute deviations (LAD) as an alternative adjustment to the commonly applied the least squares method. The LAD method is competitive, especially when observation results contain gross errors. One of the factors discouraging the application of the LAD method is its complicated algorithm, which employs the principles of linear programming. It seems that owing to its interesting properties, including the identification of observations suspected of containing gross errors, the method should gain more popularity. In this paper the proposal of LAD method will be completed with the analysis of the accuracy of obtained calculations.

2 The LAD method

The adjustment with the LAD method is done in an iterative process, employing techniques of linear programming. This study presents solutions of the LAD method given in a matrix notation. Thanks to this formula, it will be possible to determine a covariance matrix, thereby making an assessment of the accuracy of the adjustment results.
The following function of target (taking into account the weights $p_i = 1/m_i^2$) corresponds to the LAD method (VUKTOTIĆ, 1982)

$$\min_{\mathbf{X}} \psi^{LAD}(\mathbf{X}) = \sum_{i=1}^{n} p_i |v_i| = \sum_{i=1}^{n} p_i |\mathbf{A}_{S(i,\cdot)}\mathbf{X} - \mathbf{L}_i| \qquad (2.1)$$

where: v_i – element of vector of corrections, $\mathbf{L}$ – vector of absolute terms, $\mathbf{A}_S$ – known matrix of coefficients, $\mathbf{X}$ – vector of estimated model parameters, the notation $\mathbf{A}_{S(i,\cdot)}$ means an established value of the „i“ line and all the „·“ columns in the $\mathbf{A}_S$ matrix.

There is an absolute value in the function of purpose. In order to solve the problem (2.1), techniques of linear programming can be employed. The problem of linear programming is usually formulated in the following formula

$$\min_{\mathrm{x}} \Phi(\mathbf{x}) = \mathbf{c}^T\mathbf{x} \qquad (2.2)$$

with these restrictions

$$\begin{cases} \mathbf{Ax} = \mathbf{L} \\ \mathbf{x} \geq \mathbf{0} \end{cases}$$

While: **c, L, x** – columns vectors.
The function of target $\psi^{LAD}(\mathbf{X})$ should be therefore transformed so that it can be noted in a form similar to $\Phi(\mathbf{x})$. For this purpose, the vector of corrections should be shown as the difference between two non-negative quantities u_i and w_i (e.g.. VUKTOTIĆ, 1982), hence

$$v_i = u_i - w_i \quad u_i,\ wi \geq 0, \quad i = 1, 2, ..., n \qquad (2.3)$$

with

$$u_i = 0 \quad \text{for} \quad v_i \leq 0$$

$$w_i = 0 \quad \text{for} \quad v_i > 0 \qquad (2.4)$$

The function of target, described by the relationship (2.1) can now be noted as follows

$$\min_{\mathrm{x}} \psi^{LAD}(\mathbf{X}) = \sum_{i=1}^{n} p_i |v_i| = \sum_{i=1}^{n} p_i |u_i + w_i| \qquad (2.5)$$

with restrictions

$$\mathbf{v}_i = \mathbf{A}_{S(i,\cdot)}\mathbf{X} - \mathbf{L}_i \qquad (2.6)$$

or

$$\mathbf{A}_{S(i,\cdot)}\mathbf{X} - u_i + w_i = \mathbf{L}_i \quad u_i,\ w_i \geq 0 \quad i = 1, 2, ...n \qquad (2.7)$$

The optimisation problem (2.5) can be described with the following relationship

$$\sum_{j=1}^{m} 0\mathbf{X}_{\mathrm{j}} + \sum_{i=1}^{n} p_i |u_i + w_i| = \min \qquad (2.8)$$

with restrictions

$$\mathbf{A}_{S(i,\cdot)}\mathbf{X} - u_i + w_i = \mathbf{L}_i \qquad (2.9)$$

The relationships (2.8) and (2.9) are the problems of linear programming, which can be solved by for example the simplex method.

In geodetic analyses, the unknowns X_j can assume negative values, which is contrary to the concept of the simplex method. Consequently, new unknowns should be introduced to the calculations

$$\bar{X}_j = X_j + \bar{X}_{m+1} \tag{2.10}$$

with

$$\bar{X}_{m+1} = \max(0, -\min X_j) \tag{2.11}$$

Formula (2.10) means that

$$\bar{X}_{m+1} = \begin{cases} 0 & \text{when } \min X_j u < 0 \\ -\min X_j & \text{when } \min X_j > 0 \end{cases}$$

Introducing a new variable $\bar{X}_j$ results in the necessity to extend matrix $\mathbf{A}$ by an additional column $\mathbf{A}_{m+1}$. Elements $a_{(i,m+1)} = [\mathbf{A}_{m+1}]$ of the column are calculated from the following relationship

$$a_{(i,m+1)} = -\sum_{j=1}^{m} a_{i,j} \tag{2.12}$$

Taking into account the relationships (2.10) and (2.12), the optimisation problem (2.8) with restrictions (2.9) can be presented in the following final form

$$\sum_{j=1}^{m+1} 0\bar{\mathbf{X}}_j + \sum_{i=1}^{n} p_i |u_i + w_i| = \min \tag{2.13}$$

with restrictions

$$\mathbf{A}_{S(i,\cdot)}\bar{\mathbf{X}} - u_i + w_i = \mathbf{L}_i \tag{2.14}$$

where: $\bar{\mathbf{X}}, \ \mathbf{u}, \ \mathbf{v} \geq \mathbf{0}$.

The adjustment problem thus prepared will be solved by the simplex method. What follows is a general algorithm of the method in a matrix notation.

1. Seeking a basic solution
   $\mathbf{x_B} = \mathbf{h_0} = \mathbf{A_B^{-1}PL}$, ($\mathbf{x_B}$ – vector of basic variables, $\mathbf{h_0}$ – vector of basic variables values, $\mathbf{A_B^{-1}}$ – inverse of a submatrix of coefficients at the basic variables, $\mathbf{P} = \mathrm{Diag}(p_1, ..., p_n)$ – matrix of weights).
2. Checking whether the stop criterion is met, i.e. whether for the elements of vectors $\mathbf{c}$ and $\mathbf{z}$ ($\mathbf{z} = \mathbf{c_b A_B^{-1} A} = \mathbf{c_B G}$, $\mathbf{G} = \mathbf{A_B^{-1}A}$; $\mathbf{c_B}$ – subvector of vector $\mathbf{c}$ with the coefficients which correspond to the basic variables; $\mathbf{A} = [\mathbf{A}_S \vdots \mathbf{A_u} \vdots \mathbf{A_w}]$, where $\mathbf{A_u}$ and $\mathbf{A_w}$ – coefficients matrices respectively for the variables $\mathbf{u}$ and $\mathbf{w}$) which correspond to each other, the relationship $c_k - z_k \geq 0$ is satisfied.
   If **yes**, then the solution thus found is optimal.
   If **no**, the least element of vector $\mathbf{d} = \mathbf{c} - \mathbf{z}$ shall be determined. The determined element with index k, refers to a column (subvector) of matrix $\mathbf{A}$ introduced to the basic matrix $\mathbf{A_B}$ and the variable which corresponds to the subvector, which goes into the solution base $\mathbf{x_B}$.
3. Checking if in matrix $\mathbf{G}$, elements of vector $\mathbf{g}_{(\cdot,k)} \leq 0$. Then, from among the quotient of the elements of vector $\mathbf{h_0}$ by positive elements of vector $\mathbf{g}_{(\cdot,k)}$ we choose the least quotient, $\frac{h_{oi}}{g_{(\cdot,k)}} = \min\limits_{g_{k>0}} \frac{h_{0i}}{g_{(\cdot,k)}}$. Remove the variable for which the criterion is satisfied from vector $\mathbf{x_B}$, and remove the subvector corresponding to the removed basic variable from matrix $\mathbf{A_B}$.
4. Performing the calculations
   $\mathbf{h_0} = \mathbf{A_B^{-1}PL}$
   $\mathbf{d} = \mathbf{c} - \mathbf{z}$
   $\mathbf{z} = \mathbf{c_B A_B^{-1} A} = \mathbf{c_B G}$
   The value of the function of target $z_0 = \mathbf{c_B h_0}$ is determined.
5. Passing to step 2.

3 Covariance matrices

The solution of the iterative process is a basic vector $\mathbf{x_B}$ together with the values of vector $\mathbf{h_0} = \mathbf{A_B^{-1}PL}$ corresponding to those variables. In order to assess the accuracy of the thus obtained results, covariance matrices should be determined for the unknown parameters of the **Cov(X)** model, as well as for the vector of corrections **Cov(v)**. As vector $\mathbf{x_B}$ may contain both unknown parameters of model X_j and corrections v_i it is necessary to introduce the matrix of transformations $\mathbf{H}$ to the calculations of the covariance matrix **Cov(X)**. Matrix $\mathbf{H}$ is to choose only the sought parameters of the model from the vector of solutions $\mathbf{x_B}$. Let us therefore assume matrix $\mathbf{H}$ in the following form

$$\mathbf{H} = \begin{bmatrix} \frac{\partial\varphi(X_1)}{\partial x_{B1}} \frac{\partial\varphi(X_1)}{\partial x_{B2}}, \ldots, & \frac{\partial\varphi(X_1)}{\partial x_{Bn}} \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ \frac{\partial\varphi(X_m)}{\partial x_{B1}} \frac{\partial\varphi(X_m]}{\partial x_{B2}}, \ldots, & \frac{\partial\varphi(X_m)}{\partial x_{Bn}} \end{bmatrix} \tag{3.1}$$

where $X_j = \bar{X}_j - \bar{X}_{m+1}$ and x_{Bi} – variables contained in vector $\mathbf{x_B}$ of the final solution.

It can be therefore written

$$\mathbf{X} = \mathbf{H}\ \mathbf{A_B^{-1}PL} \tag{3.2}$$

Employing the principle of variance transfer, which is known to the literature of the subject and is usually formulated as follows

$$\mathbf{Cov} = \mathbf{DC}(\mathbf{L})\mathbf{D}^{\mathrm{T}} \tag{3.3}$$

where: **Cov** – sought covariance matrix, $\mathbf{D}$ – matrix of known coefficients, $\mathbf{C_L}$ – known matrix of covariance of measurement results.

Hence the covariance matrix of a vector of parameters **Cov(X)** assumes the following form (with $\mathbf{D} = \mathbf{H}\ \mathbf{A_B^{-1}P}$)

$$\mathbf{Cov}(\mathbf{X}) = \mathbf{D}\ \mathbf{C}(\mathbf{L})\mathbf{D}^{\mathrm{T}} = (\mathbf{H}\ \mathbf{A_B^{-1}P})\mathbf{C}(\mathbf{L})(\mathbf{P}(\mathbf{A_B^{-1}})^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}) \tag{3.4}$$

Taking into account that

$$\mathbf{C}(\mathbf{L}) = m_0^2\mathbf{P}^{-1} \tag{3.5}$$

where, m_0^2 – estimator of coefficient of variance

$$\mathbf{Cov}(\mathbf{X}) = m_0^2\mathbf{H}\ \mathbf{A_B^{-1}P}\ \mathbf{P}^{-1}\ \mathbf{P}(\mathbf{A_B^{-1}})^{\mathrm{T}}\mathbf{H}^{\mathrm{T}} \tag{3.6}$$

The relationship (3.6) can also be written in the following form

$$\mathbf{Cov}(\mathbf{X}) = m_0^2\mathbf{Q}(\mathbf{X}) \tag{3.7}$$

where

$$\mathbf{Q}(\mathbf{X}) = \mathbf{H}\ \mathbf{A_B^{-1}P}(\mathbf{A_B^{-1}})^{\mathrm{T}}\mathbf{H}^{\mathrm{T}} \tag{3.8}$$

is a matrix of co-factors (approximations of variance) of a vector of parameters **X**.
Coefficient m_0^2 can be determined from a well-known relationship $m_0^2 = \mathbf{v}^T\mathbf{P}\mathbf{v}/(n-m)$.
In order to determine the covariance matrix of a vector of corrections **v**, the following relationship can be formulated

$$\mathbf{V} = \mathbf{A}_S\mathbf{X} - \mathbf{L} = \mathbf{A}_S\mathbf{H}\ \mathbf{A}_\mathbf{B}^{-1}\mathbf{P}\mathbf{L} - \mathbf{L} = -(\mathbf{E} - \mathbf{A}_S\mathbf{H}\ \mathbf{A}_\mathbf{B}^{-1}\mathbf{P})\mathbf{L} \tag{3.9}$$

where $\mathbf{E} = \mathrm{diag}(1, ..., 1)$.
Hence, assuming that $\mathbf{Cov} = \mathbf{DC(L)D}^{\mathrm{T})}$ the following results

$$\mathbf{Cov}(\mathbf{v}) = (\mathbf{E} - \mathbf{A}_S\mathbf{H}\ \mathbf{A}_\mathbf{B}^{-1}\mathbf{P})\mathbf{C}(\mathbf{L})(\mathbf{E} - \mathbf{A}_S\mathbf{H}\ \mathbf{A}_\mathbf{B}^{-1}\mathbf{P})^{\mathrm{T}} \tag{3.10}$$

where

$$\mathbf{D} = \mathbf{E} - \mathbf{A}_S\mathbf{H}\ \mathbf{A}_\mathbf{B}^{-1}\mathbf{P} \tag{3.11}$$

Having performed the calculations (with $\mathbf{C}(\mathbf{L}) = \mathrm{m}_0^2\mathbf{P}^{-1}$) the covariance matrix of a vector of corrections is obtained in the following form

$$\mathbf{Cov}(\mathbf{v}) = \mathrm{m}_0^2(\mathbf{P}^{-1} - (\mathbf{A}_\mathbf{B}^{-1})^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{A}_S^{\mathrm{T}} - \mathbf{A}_S\mathbf{H}\ \mathbf{A}_\mathbf{B}^{-1} + \mathbf{A}_S\mathbf{H}\ \mathbf{A}_\mathbf{B}^{-1}\mathbf{P}(\mathbf{A}_\mathbf{B}^{-1})^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}\ \mathbf{A}_S^{\mathrm{T}}) \tag{3.12}$$

or

$$\mathbf{Cov}(\mathbf{v}) = \mathrm{m}_0^2(\mathbf{P}^{-1} - (\mathbf{A}_\mathbf{B}^{-1})^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{A}_S^{\mathrm{T}} - \mathbf{A}_S\mathbf{H}\ \mathbf{A}_\mathbf{B}^{-1} + \mathbf{A}_S\mathbf{Q}(\mathbf{X})\mathbf{A}_S^{\mathrm{T}}) \tag{3.13}$$

where $\mathbf{Q}(\mathbf{X}) = \mathbf{H}\ \mathbf{A}_\mathbf{B}^{-1}\mathbf{P}(\mathbf{A}_\mathbf{B}^{-1})^{\mathrm{T}}\ \mathbf{H}^{\mathrm{T}}$.
Assuming that a cofactor matrix **Q(v)** is formulated follows

$$\mathbf{Q}(\mathbf{v}) = (\mathbf{P}^{-1} - (\mathbf{A}_\mathbf{B}^{-1})^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{A}_S^{\mathrm{T}} - \mathbf{A}_S\mathbf{H}\ \mathbf{A}_\mathbf{B}^{-1} + \mathbf{A}_S\mathbf{Q}(\mathbf{X})\mathbf{A}_S^{\mathrm{T}}) \tag{3.14}$$

The covariance matrix of a vector of corrections has the following form

$$\mathbf{Cov}(\mathbf{v}) = \mathrm{m}_0^2\mathbf{Q}(\mathbf{v}) \tag{3.15}$$

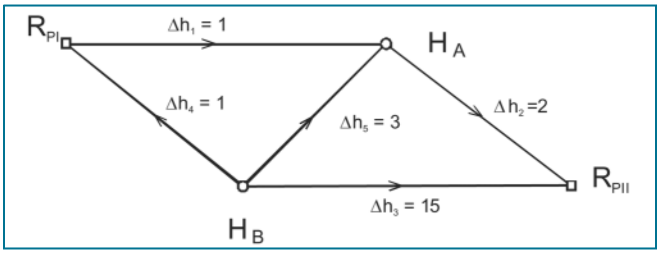

Fig. 1

4 An example of practical application

In order to present the LAD method in practical terms, we present a simulated example of a levelling network. The analysed network is presented in Fig 1. The figure assumes that $\mathrm{R_{PI}}$, $\mathrm{R_{PII}}$ – points of reference, $\mathrm{H_A}$, $\mathrm{H_B}$ – points of determining altitudes, Δh – „measured overheight“. In order to simplify the calculations it was assumed that the network lies on a plane and points altitudes equal 0 (in any units). A gross error is meant to denote the overheight $\Delta h_3 = 15$.
The function of target can be formulated as follows:

$$\sum_{j=1}^{m} 0X_j + \sum_{i=1}^{n} p_i|u_i + w_i| = \min \rightarrow \mathbf{c}\mathbf{x}^T = \underbrace{[00\vdots 0\vdots 11111\vdots 11111]}_{\mathbf{c}}\underbrace{[H_A H_B \vdots \bar{H}_3 \vdots u_1u_2u_3u_4u_5 \vdots w_1w_2w_3w_4w_5]^T}_{\mathbf{x}^T}$$

with restrictions

$$\begin{array}{llll}
\mathrm{H_A} & & -\bar{\mathrm{H}}_3 - u_1 + w_1 = \Delta h_1 \\
-\mathrm{H_A} & & +\bar{H}_3 - u_2 + 2_2 = \Delta h_2 \\
 & -\mathrm{H_B} & +\bar{\mathrm{H}}_3 - u_3 + w_3 = \Delta h_3 \\
 & -\mathrm{H_B} & +\bar{H}_3 - u_4 + w_4 = \Delta h_4 \\
\mathrm{H_A} & -\mathrm{H_B} & \quad - u5 + w_5 = \Delta h_5
\end{array}$$

where $\mathbf{A}_S^{\mathrm{T}} = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 \end{bmatrix}^{\mathrm{T}}$

The restrictions can also be presented in the following formula

$$\underbrace{\left[\begin{array}{cc:c:ccccc:ccccc}
1 & 0 & -1 & -1 & & & & & 1 & & & & \\
-1 & 0 & 1 & & -1 & & & & & 1 & & & \\
0 & -1 & 1 & & & -1 & & & & & 1 & & \\
0 & -1 & 1 & & & & -1 & & & & & 1 & \\
1 & -1 & 0 & & & & & -1 & & & & & 1
\end{array}\right]}_{\mathbf{A}}
\underbrace{\begin{bmatrix} \mathrm{H_A} \\ \mathrm{H_B} \\ \bar{\mathrm{H}}_3 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ \cdots \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{bmatrix}}_{\mathbf{x}}
= \underbrace{\begin{bmatrix} \Delta h_1 \\ \Delta h_2 \\ \Delta h_3 \\ \Delta h_4 \\ \Delta h_5 \end{bmatrix}}_{\mathbf{L}}$$

Generally, it can be written that $\mathbf{A}\mathbf{x} = \mathbf{L}$. According to the principles of a simplex method, a basic solution is sought. For further considerations let us assume a transformed matrix $\mathbf{A} = [\mathbf{A_B} \vdots \mathbf{A_P}]$, in which $\mathbf{A_B}$ is a matrix of coefficients at the basic variables $\mathbf{x_B}$, while $\mathbf{A_P}$ is a matrix of coefficients of non-basic variables $\mathbf{x_P}$, $(\mathbf{x} = [\mathbf{x_B} \vdots \mathbf{x_P}]^T)$.
Taking into account a matrix of weights $\mathbf{P}$, the following formula can be written

$$[\mathbf{A_B} \vdots \mathbf{A_P}]\mathbf{P}\begin{bmatrix} \mathbf{x_B} \\ \dots \\ \mathbf{x_P} \end{bmatrix} = \begin{bmatrix} \mathbf{L} \\ \dots \\ \mathbf{0} \end{bmatrix}$$

hence

$$\mathbf{A_B P x B} + \mathbf{A_P P x_P} = \mathbf{L} \rightarrow \mathbf{x_B} = \mathbf{A_B^{-1} P}(\mathbf{L} - \mathbf{A_P P x_P})$$

Assuming that $\mathbf{x_P} = \mathbf{0}$, $(\mathrm{H_A} = \mathrm{H_B} = \bar{\mathrm{H}}_3 = u_1 = u_2 = u_3 = u_4 = u_5 = 0)$
we have

$$\mathbf{x_B} = \mathbf{h_0} = \mathbf{A_B^{-1} P L},$$

Therefore (assuming that a matrix of weights $\mathbf{P} = \mathrm{Diag}(1, ...1)$)

$$\underbrace{\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{bmatrix}}_{\mathbf{x_B}} = \underbrace{\left(\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}\right)^{-1}}_{\mathbf{A_B^{-1}}} \underbrace{\begin{bmatrix} 1 \\ 2 \\ 15 \\ 1 \\ 3 \end{bmatrix}}_{\mathbf{L}} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 15 \\ 1 \\ 3 \end{bmatrix}}_{\mathbf{h_0}}$$

For further calculations we assume the following (altered) form of matrix $\mathbf{A}$

$$\mathbf{A} = \left[\begin{array}{ccccc:cc:c:ccccc} 1 & & & & & 1 & 0 & -1 & -1 & & & & \\ & 1 & & & & -1 & 0 & 1 & & -1 & & & \\ & & 1 & & & 0 & -1 & 1 & & & -1 & & \\ & & & 1 & & 0 & -1 & 1 & & & & -1 & \\ & & & & -1 & 1 & -1 & 0 & & & & & -1 \end{array}\right],$$

$$\mathbf{A_B} \qquad\qquad \mathbf{A_P}$$

Performing calculations in accordance with the algorithm presented above, the following solution has been achieved after three iterations: $\bar{\mathrm{H}}_A = 3$, $w_2 = 3$, $w_3 = 13$, $\bar{H} = 2$, $u_4 = 1$.
Applying the notation introduced earlier $(\mathrm{X_j} = \bar{\mathrm{X}}_j - \bar{\mathrm{X}}_3)$, the unknown parameters can be determined, namely

$$\mathrm{H_A} = \bar{\mathrm{H}}_A - \bar{H}_3 = 3 - 2 = 1$$

$$\mathrm{H_B} = \bar{\mathrm{H}}_B - \bar{H}_3 = 0 - 2 = -2$$

as well as the vector of corrections $v_i = u_i - w_i$ (with $u_i = 0$ for $v_i \leq 0$, and $w_i = 0$ for $v_i > 0$)
$v_i = 0$ (u_1 and w_1 are not in the solution),
$v_2 = -3$ $w_2 = 3$ ($u_2 = 0$ to v_2 is negative),
$v_3 = -13$ $w_3 = 13$ ($u_3 = 0$ to v_2 is negative),
$v_4 = 1$ $u_4 = 4$ ($u_4 = 0$ to v_4 is positive)
$v_5 = 0$ (u_5 and w_5 are not in the solution).
Based on the final results of adjustment, it can be noted that $v_3 = -13$. It is therefore highly probable that the observation contains a gross error.
The following results were obtained in III iteration

$$\mathbf{G} = \underbrace{\left(\begin{bmatrix} 1 & & & -1 & 0 \\ -1 & 1 & & 1 & 0 \\ 0 & & 1 & 1 & 0 \\ 0 & & & 1 & -1 \\ 1 & & & 0 & 0 \end{bmatrix}\right)^{-1}}_{\mathbf{A_B^{-1}}} \underbrace{\left[\begin{array}{ccccc:cc:c:ccccc} 1 & & & -1 & 0 & 1 & -1 & 0 & -1 & & & & \\ -1 & 1 & & 1 & 0 & -1 & 1 & 0 & & -1 & & & \\ 0 & & 1 & 1 & 0 & 0 & 0 & 0 & & & -1 & & \\ 0 & & & 1 & -1 & 0 & -1 & 1 & & & & -1 & \\ 1 & & & 0 & 0 & 1 & -1 & 0 & & & & & -1 \end{array}\right]}_{\mathbf{A}}$$

hence

$$\mathbf{G} = \begin{bmatrix} 1 & & & & & \vdots & 1 & -1 & \vdots & 0 & \vdots & 0 & 0 & 0 & 0 & -1 \\ & 1 & & & & \vdots & 0 & 0 & \vdots & 0 & \vdots & -1 & -1 & 0 & 0 & 0 \\ & & 1 & & & \vdots & 0 & 0 & \vdots & 0 & \vdots & -1 & 0 & -1 & 0 & 1 \\ & & & 1 & & \vdots & 0 & -1 & \vdots & 1 & \vdots & 1 & 0 & 0 & 0 & -1 \\ & & & & 1 & \vdots & 0 & 0 & \vdots & 0 & \vdots & 1 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\mathbf{h_0} = \mathbf{A_B^{-1}PL}$$

$$\mathbf{h_0} = \left(\begin{bmatrix} 1 & & & -1 & 0 \\ -1 & 1 & & 1 & 0 \\ 0 & & 1 & 1 & 0 \\ 0 & & & 1 & -1 \\ 1 & & & 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 2 \\ 15 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 13 \\ 2 \\ 1 \end{bmatrix}$$

$$\mathbf{z} = \mathbf{c_B G} \qquad \mathbf{c_B} = \begin{matrix} [0 & 1 & 1 & 0 & 1] \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ H_A & w_2 & w_3 & \bar{H}_3 & u_4 \end{matrix}$$

$$\mathbf{z} = [0\ 1\ 1\ 0\ 1 \vdots 0\ 0 \vdots 0 \vdots -1\ \ -1\ \ -1\ 1\ 0]$$

$$\mathbf{d} = \mathbf{c} - \mathbf{z}$$

$$\mathbf{d} = [[1\ 1\ 1\ 1\ 1 \vdots 0\ 0 \vdots 0 \vdots 1\ 1\ 1\ 1\ 1] - [0\ 1\ 1\ 0\ 1 \vdots 0\ 0 \vdots 0 \vdots -1\ \ -1\ \ -1\ 1\ 0]$$

$$\mathbf{d} = [1\ 0\ 0\ 1\ 0 \vdots 0\ 0 \vdots 0 \vdots 2\ 2\ 2\ 0\ 1]$$

$$z_0 = \mathbf{c_B h_0} = [0\ 1\ 1\ 0\ 1] \begin{bmatrix} 3 \\ 3 \\ 13 \\ 2 \\ 1 \end{bmatrix} = 17$$

Subsequently, an analysis of accuracy is performed with the use of the relationships derived earlier. Let us assume matrix $\mathbf{A_B}$, obtained from the final solution and equal

$$\mathbf{A_B} = \begin{bmatrix} 1 & & & -1 & 0 \\ -1 & 1 & & 1 & 0 \\ 0 & & 1 & 1 & 0 \\ 0 & & & 1 & -1 \\ 1 & & & 0 & 0 \end{bmatrix}$$

In order to calculate a covariance matrix $\mathbf{Cov}(\mathbf{X}) = m_0^2\mathbf{Q}(\mathbf{X})$ of a vector of parameters (where matrix $\mathbf{Q(X)}$ is described by the relationship (3.8)), in the analysed example, we can write

$$H_A = \bar{H}_A - \bar{H}_3$$

$$H_B = \bar{H}_B - \bar{H}_3$$

Hence;

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial H_A}{\partial \bar{H}_A} & \dfrac{\partial H_A}{\partial w_2} & \dfrac{\partial H_A}{\partial w_3} & \dfrac{\partial H_A}{\partial \bar{H}_3} & \dfrac{\partial H_A}{\partial u_4} \\ \dfrac{\partial H_B}{\partial \bar{H}_A} & \dfrac{\partial H_B}{\partial w_2} & \dfrac{\partial H_B}{\partial w_3} & \dfrac{\partial H_B}{\partial \bar{H}_3} & \dfrac{\partial H_B}{\partial u_4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

Therefore;

$$\mathbf{Cov}(\mathbf{X}) = \mathrm{m}_0^2 \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 & -1 \\ -1 & 1 & 3 & -2 & -2 \\ 1 & -1 & -2 & 2 & 2 \\ 1 & -1 & -2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & -1 \\ 0 & 0 \end{bmatrix}$$

Hence;

$$\mathbf{Cov}(\mathbf{X}) = \mathrm{m}_0^2 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

A covariance matrix of a vector of corrections described by the relationship $\mathbf{Cov}(\mathbf{v}) = \mathrm{m}_0^2\mathbf{Q}(\mathbf{v})$, for $\mathbf{Q}(\mathbf{v}) = (\mathbf{P}^{-1} - (\mathbf{A}_\mathbf{B}^{-1})^\mathrm{T}\mathbf{H}^\mathrm{T}\mathbf{A}_S^\mathrm{T} - \mathbf{A}\ \mathbf{H}\ \mathbf{A}_\mathbf{B}^{-1} + \mathbf{A}_S\mathbf{Q}(\mathbf{X})\mathbf{A}_S^\mathrm{T})$ will have the following form

$$\mathbf{Cov}(\mathbf{v}) = \mathrm{m}_0^2 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In covariance matrix **Cov (v)** there are no mean errors of corrections v_1 and v_5. The mean of both corrections can be determined following (on the basis of fig. 1)

$$\Delta h_1 + v_1 = H_A - R_{PI} \qquad v_1 = H_A - R_{PI} - \Delta h_1$$

hence

$$\Delta h_5 + v_5 = H_A - H_B \qquad v_5 = H_A - H_B - \Delta h_5$$

Hence (assuming the error free points of reference)

$$m_{v_1}^2 = \left(\frac{\partial v_1}{\partial H_A}\right)^2 m_{H_A}^2 + \left(\frac{\partial v_1}{\partial \Delta h_1}\right)^2 m_{\Delta h_1}^2 = 1^2 + 1^2 = 2; \qquad m_{v_1} = \sqrt{2} = 1{,}4$$

$$m_{v_5}^2 = \left(\frac{\partial v_5}{\partial H_A}\right)^2 m_{H_A}^2 + \left(\frac{\partial v_5}{\partial H_B}\right)^2 m_{H_B}^2 + \left(\frac{\partial v_1}{\partial \Delta h_5}\right)^2 m_{\Delta h_5}^2 = 1^2 + 2^2 + 1^2 = 6; \quad m_{v_5} = \sqrt{6} = 2{,}4$$

5 Summary

This paper presents the problem of robust adjustment with an analysis of results accuracy by the method of the least absolute deviations (LAD).
The forms of covariance matrices of a vector of parameters are presented as well as corrections, which provide an assessment of accuracy of the final determinations.

Literature

[1] Vuktotić, N. 1982: Lineare Programmierung, Deutsche Geodätische Kommission bei der Bayerischen Akademie der Wissenschaften. Reihe A, Heft Nr. 94, München

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