Establishing of a damping function criterion in the robust adjustment algorithm

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The paper proposes a new way of standardising observation adjustments and a way of establishing a damping function criteria, applied in algorithms of adjustment insensitive to gross errors. It presents theoretical formulae and practical verification on a numerical example. The effectiveness of the proposed method has been assessed by comparison of the results with the classic (known) methods of standardisation and determination of damping function parameters.

1 Introduction and formulation of the problem

The concept of adjustment insensitive to gross errors has been put forward by Huber [5]. It was subsequently developed by Hampel [4] and other scholars (cf. e.g. [14]). Similar subjects have been dealt with in a number of other reports, e.g. [1, 10, 6–8, 9, 12] and other. A detailed description of selected methods of insensitive adjustment can be found in [11, 13] and [14].

One of the methods of error-insensitive adjustment of geodetic observations is a modified least square method. The modification comes down to replacing the traditional weighted matrix with a matrix function of weights, based on the so called damping function. The aim of the function is to compensate (damp) the effect of deviating observations (gross errors) on the final results of adjustment. Among the known damping functions, there are the following: Huber's function, Hampel's function, Danish function [13] as well as the functions proposed by the author of this paper [2, 3].

A significant drawback of the group of error-insensitive adjustment methods, based on the application of a damping function, is the necessity to standardise observation adjustments and to establish so called criteria (controlling parameters) of a damping function. Both the processes are quite complicated and there are no clearly defined criteria of (empiric) selection of the controlling parameters [11].

The values of standardised adjustments $\bar{v}_i$ are calculated from adjustments v_i, estimated by the classic method of least squares (*LSQ*):

$$\bar{v}_i = \frac{v_i}{m_{v_i}} \tag{1}$$

Errors of mean adjustments m_{v_i} are square roots of diagonal elements of a covariance matrix of a vector of observation adjustments:

$$\mathbf{Q}_V = \mathbf{P}^{-1} - \mathbf{A}(\mathbf{A}^T\mathbf{P}\mathbf{A})^{-1}\mathbf{A}^T \tag{2}$$

Symbols **A, P** denote matrices which are present in a classic adjustment according to *LSQ* (**A** – a matrix of factors with unknown parameters in equations of errors, **P** – matrix of weights). Formula (2) is correct with the assumption that the mean unit error (standard deviation estimator) $m_0 = 1$.

Thus calculated standardised adjustments (1) are verified with a damping function (e.g. *QDF* – see [2]):

$$f(\bar{v}) = \begin{cases} 1; & \bar{v} \in \langle -k_0; k_0 \rangle \\ 1 - \frac{\bar{v}^2 - 2k_0|\bar{v}| + k_0^2}{(k-k_0)^2}; & |\bar{v}| \in \langle k_0; k \rangle \\ 0; & |\bar{v}| > k \end{cases} \tag{3}$$

Examples of damping functions are shown in Fig. 1.

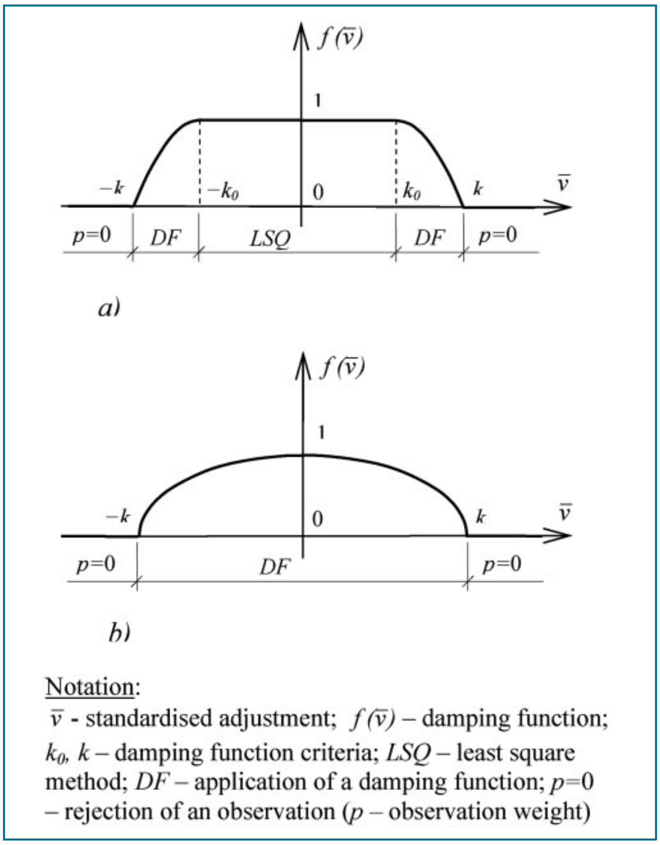

Fig. 1:* *Examples of damping functions: a) QDF – see [2], b) EDF – see [3]

A criterion of a damping function (controlling parameter) k_0, present in equation (3), is selected for the assumed normal distribution of observation errors [13]. The first step is to calculate the level of probability γ for the determined interval of standardised adjustments $\Delta\bar{v} = -k_0; k_0$. The value of parameter k_0 is then determined:

$$P(\bar{v} \in k_0; k_0) = 2\int_0^{k_0} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-\bar{v}^2}{2}\right) d\bar{v} = 2\phi(k_0) = \gamma \tag{4}$$

The value of function $\phi(k_0)$ is read from the tables of normal distribution for argument k_0. The second criterion of the damping function (parameter k – cf. Fig. 1) is adopted empirically. A too low value of the parameter may result in a risk of failure to notice a deviating observation, whereas a too high one – may result in too slow convergence of the iterative process. It is recommended that the initial value of parameter k should be taken from the interval $\langle 4;\, 6\rangle$. The aim of this study is to put forward a proposition of a simplified method of adjustment standardisation and a new principle of establishing damping function criteria.

2 Standardisation of adjustments and damping function criteria

Following is a proposed sequence of calculations of an error-insensitive adjustment using any of the known damping functions (see, e.g. [13]):

1) Calculation of adjustments according to v_i algorithm based on the least square condition (e.g. by the parametric method).
2) Standardisation of adjustments ($v_i \rightarrow \bar{v}_i$):

$$\bar{v}_i = \frac{v_i}{m_i} \tag{5}$$

where: m_i – mean error of the i-th observation.

3) Calculation of the arithmetic average $|\bar{v}|_{ar}$ of absolute values of standardised adjustments:

$$|\bar{v}|_{ar} = \frac{\Sigma|\bar{v}_i|}{n} \tag{6}$$

where: n – the number of observations in a group of uniform observations.

4) Calculations of deviations $\delta\bar{v}_i$ of standardised adjustments from the mean value:

$$\delta\bar{v}_i = |\bar{v}_i| - |\bar{v}|_{ar} \tag{7}$$

5) Determination of a deviation coefficient μ_i for each standardised adjustment:

$$\mu_i = \frac{d\bar{v}_i}{|\bar{v}|_{ar}} \tag{8}$$

6) Application of coefficient μ_i as an auxiliary criterion of a damping function:

$$\begin{cases} \mu_i \leq k_0 & \rightarrow f(\bar{v}) = 1, (LSQ) \\ k_0 < \mu_i < k & \rightarrow f(\bar{v}) \in (0; 1) \\ \mu_i \geq k & \rightarrow f(\bar{v}) = 0, (p = 0) \end{cases} \tag{9}$$

where: $f(\bar{v})$ – damping function; k_0, k – criteria of application of a damping function; p – weight of an observation in performing adjustment with the least squares method (LSQ).

The formula (8) can be written in the following form:

$$\mu_i = n \times \frac{|\bar{v}_i|}{|\Sigma v_i|} - 1 \tag{10}$$

Substituting $|\bar{v}_i| = k_0$ and $|\bar{v}_i| = k$ in formula (10), it can be established for what values of the standardised adjustment $\bar{v}_i$ a damping function should be applied (criterion k_0) or a given observation should be excluded from the adjustment process (criterion k). Transformation of formula (10) yields:

$$k = (\mu_{\max} + 1) \times |\bar{v}|_{ar} \tag{11}$$

$$k_0 = (\mu_0 + 1) \times |\bar{v}|_{ar} \tag{12}$$

The values μ_0 and $\mu_{\max}$ are adopted from formula (8). For example, assuming that

$$\mu_0 = -0.5 \text{ and } \mu_{\max} = 0.5 \tag{13}$$

formulae (11) and (12) can be written in a simpler manner:

$$k = 1.5 \times |\bar{v}|_{ar} \tag{14}$$

$$k_0 = 0.5 \times |\bar{v}|_{ar} \tag{15}$$

It seems that the restricting values of the absolute coefficient of a standardised adjustment ($\mu_0, \mu_{\max}$), adopted for formulae (14), (15), are optimal. The conviction stems from the following reasoning. It is easy to prove that the mean for the value calculated from formula (8) or (10) is equal to zero:

$$\mu_{ar} = 0 \tag{16}$$

It is a neutral point, so the restricting values should be situated symmetrically in relation to the point (16). On the other hand, it is noteworthy that the lower restricting value of μ_i is „– 1“, which happens when $\bar{v}_i = 0$:

$$\lim_{v_{\mathrm{i}} \to 0} \mu_{\mathrm{i}} = -1 \tag{17}$$

Therefore it seems natural that the lower limit of damping should be adopted in the middle of the interval (– 1;0), i.e. $\mu_0 = -0.5$.
Adopting lower values ($\mu_0 < -0.5, \mu_{\max} < 0.5$) can bring about unnecessary damping or rejecting correct observation, respectively. Whereas higher values of the coefficients ($\mu_0 > -0.5$, $\mu_{\max} > 0.5$) may result in that, that an observation with a gross error will greatly affect the results of adjustment.
Another way to determine an auxiliary criterion of a damping function (8), (10) is to calculate (instead of the arithmetic mean of the absolute values $|\bar{v}|_{\mathrm{ar}}$ (6)) the mean value of the sum of squares of standardised adjustments $\bar{v}_i$:

$$(\bar{v}^2)_{ar} = \frac{\Sigma \bar{v}_i^2}{n} \tag{18}$$

An auxiliary criterion of a damping function is then determined in the following manner:

$$m_i = n \times \frac{\bar{v}_i^2}{\Sigma \bar{v}_i^2} - 1 \tag{19}$$

Applying the same procedure to formulae (7), (8), (10), yields alternative formulae for the criteria of damping functions (11), (12):

$$k = \sqrt{(\mu_{\max} + 1) \times (\bar{v}^2)_{ar}} \tag{20}$$

$$k_0 = \sqrt{(\mu_0 + 1) \times (\bar{v}^2)_{ar}} \tag{21}$$

As in this case the conclusions from formulae (16) and (17) are also true, the restricting values μ_0, $\mu_{\max}$ are adopted like before (13). The formulae equivalents to (14) and (15), when (18) is applied, will be written in the following manner:

$$k = \sqrt{1.5 \times (\bar{v}^2)_{ar}} \tag{22}$$

$$k_0 = \sqrt{0.5 \times (\bar{v}^2)_{ar}} \tag{23}$$

The formulae for calculation of an auxiliary criterion of a damping function (10), (19) will be illustrated with an example based on theoretical values. Table 1 contains two sets of standardised adjustments (A – with gross errors, B – without gross errors) with corresponding coefficients μ. For „version I“ formula (10) has been applied (proposition I), and for „version II“ – formula (19) (proposition II). Fig. 2 and 3 show graphical illustrations of data contained in Table 1. The illustrations indicate that only two deviat-

Table 1: The values of an auxiliary criterion of damping, calculated for example sets of standardised adjustments (corrections).

No.	Version I				Version II			
	A		B		A		B	
	$\bar{v}$	μ	$\bar{v}$	μ	$\bar{v}$	μ	$\bar{v}$	μ
1	**– 20**	**1.97**	–	–	**– 20**	**4.13**	–	–
2	– 10	0.48	– 10	0.91	– 10	0.28	– 10	1.73
b	– 9	0.34	– 9	0.72	– 9	0.04	– 9	1.21
4	– 8	0.19	– 8	0.53	– 8	– 0,18	– 8	0.75
5	– 7	0.04	– 7	0.34	– 7	– 0.37	– 7	0.34
6	– 6	– 0.1	– 6	0.15	– 6	– 0.54	– 6	– 0.02
7	– 5	– 0.26	– 5	– 0.05	– 5	– 0.68	– 5	– 0.32
8	– 4	– 0.41	– 4	– 0.24	– 4	– 0.79	– 4	– 0.56
9	– 3	– 0.55	– 3	– 0.43	– 3	– 0.88	– 3	– 0.75
10	– 2	– 0.70	– 2	– 0.62	– 2	– 0.95	– 2	– 0.89
11	– 1	– 0.85	– 1	– 0.81	– 1	– 0.99	– 1	– 0.97
12	0	– 1.00	0	– 1.00	0	– 1.00	0	– 1.00
13	1	– 0.85	1	– 0.81	1	– 0.99	1	– 0.97
14	2	– 0.70	2	– 0.62	2	– 0.95	2	– 0.89
15	3	– 0.55	3	– 0.43	3	– 0.88	3	– 0.75
16	4	– 0.41	4	– 0.24	4	– 0.79	4	– 0.56
17	5	– 0.26	5	– 0.05	5	– 0.68	5	– 0.32
18	6	– 0,11	6	0.15	6	– 0.54	6	– 0.02
19	7	0.04	7	0.34	7	– 0.37	7	0.34
20	8	0.19	8	0.53	8	– 0.18	8	0.75
21	9	0.34	9	0.72	9	0.04	9	1.21
22	10	0.48	10	0.91	10	0.28	10	1.73
23	**25**	**2.71**	–	–	**25**	**7.01**	–	–

Notation:
Version I– see formula (10); Version II – see formula (19);
A – a set of observations with gross errors;
B – a set of observations without gross errors;
$\bar{v}$ – standardised adjustment;
μ – an auxiliary criterion of damping

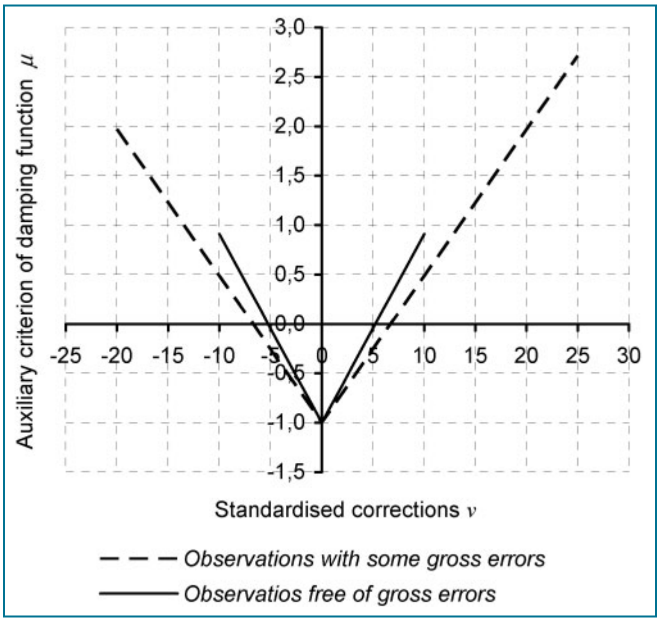

***Fig. 2:** An auxiliary criterion of a damping function (example) – version I*

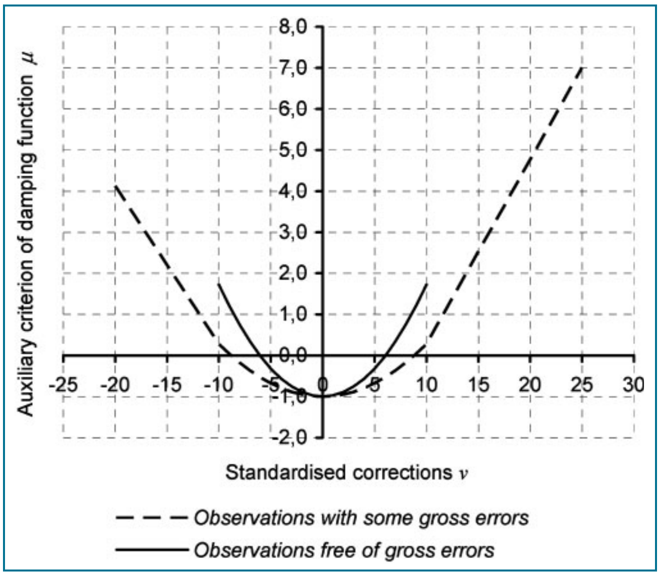

***Fig. 3:** An auxiliary criterion of a damping function (example) – version II*

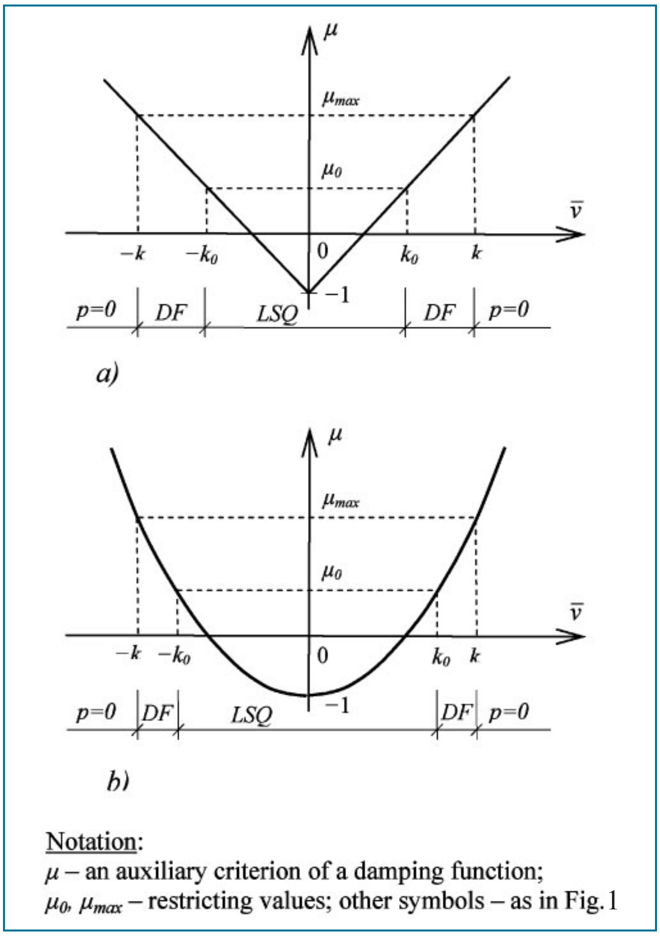

***Fig. 4:** An auxiliary criterion of a damping function: a) version I – cf. (10), b) version II – cf. (19)*

ing observations significantly change the diagrams of the proposed coefficient, which will hopefully result in its application in detecting gross errors.

Diagrams in Fig. 4 show the relationships between coefficient μ and the controlling parameters (k_0, k) of a damping function. Judging from the diagram shapes, proposition II (fig. 4b) is more beneficial: it protects observations with low values of standardised adjustments and is more restrictive towards the observations with higher values of attributed adjustments (contaminated observations).

3 A numerical example

A numerical example will be shown on the database, containing data which were used in an earlier study by the author [2]. A set of observations contains equally accurate results of 4 measurements of a certain length x: $d_i = \{100.006;\ 100.003;\ 99.997;\ 100.054\}$, one of which significantly deviates from the others. The approximate value of the unknown has been adopted to be equal to $x_0 = 100.000$ m, whereas the mean error of the measurement is equal to $m_0 = 0.005$ m.

In order to compare which of the methods yields more credible results, an adjustment will also be performed according to classic *LSQ*. Here, the set of observations does not contain any gross errors ($d_i = \{100.006;\ 100.003;\ 99.997;\ 100.002\}$). Apparently, the deviating observation d_4 has been replaced with the expected value (arithmetic average) of the remaining three observations.

The example makes use of the formulae for two damping functions: QDF (3) and Hampel's – cf. e.g. [13]:

$$f(\bar{v}) = \frac{|\bar{v}| - k}{k_0 - k} \tag{24}$$

The calculations and results of the adjustment performed by various methods (in several versions) are presented in Table 2. A detailed description of symbols and versions of

Table 2: Results of adjustment for different versions

Quantities being calculated	No.	Classic version		Proposed solutions								LSQ*
				Version I		Version II		Version I*		Version II*		
		Hampel	QDF	Hampel	QDF	Hampel	QDF	Hampel	QDF	Hampel	QDF	
v_i [mm]	1	9.0	9.0	9.0	9.0	9.0	9.0	9.0	9.0	9.0	9.0	– 4.0
	2	12.0	12.0	12.0	12.0	12.0	12.0	12.0	12.0	12.0	12.0	– 1.0
	3	18.0	18.0	18.0	18.0	18.0	18.0	18.0	18.0	18.0	18.0	5.0
	4	**– 39.0**	**– 39.0**	**– 39.0**	**– 39.0**	**– 39.0**	**– 39.0**	**– 39.0**	**– 39.0**	**– 39.0**	**– 39.0**	0.0
δx [mm]	–	15.0	15.0	15.0	15.0	15.0	15.0	15.0	15.0	15.0	15.0	2.0
$\bar{v}_i$ [mm]	1	2.08	2.08	1.80	1.80	1.80	1.80	2.08	2.08	2.08	2.08	–
	2	2.77	2.77	2.40	2.40	2.40	2.40	2.77	2.77	2.77	2.77	
	3	4.16	4.16	3.60	3.60	3.60	3.60	4.16	4.16	4.16	4.16	
	4	– 9.01	– 9.01	– 7.80	– 7.80	– 7.80	– 7.80	– 9.01	– 9.01	– 9.01	– 9.01	
μ	1	–	–	– 0.54	– 0.54	– 0.84	– 0.84	– 0.54	– 0.54	– 0.84	– 0.84	–
	2			– 0.38	– 0.38	– 0.72	– 0.72	– 0.39	– 0.39	– 0.72	– 0.72	
	3			– 0.88	– 0.88	– 0.37	– 0.37	– 0.88	– 0.88	– 0.37	– 0.37	
	4			1.00	1.00	1.94	1.94	1.00	1.00	1.94	1.94	
μ_0	–	–	–	– 0.5	– 0.5	– 0.5	– 0.5	– 0.5	– 0.5	– 0.5	– 0.5	–
$\mu_{\max}$	–	–	–	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	–
k_0	–	2.0	2.0	1.85	1.85	3.22	3.22	2.25	2.25	3.71	3.71	–
k	–	6.0	6.0	5.85	5.85	5.58	5.58	6.76	6.76	6.44	6.44	–
$v_i^{(1)}$ [mm]	1	– 2.92	– 3.47	– 3.15	– 3.66	– 3.72	– 3.95	– 3.19	– 3.68	– 3.72	– 3.95	**– 4.0**
	2	0.08	– 0.47	– 0.15	– 0.66	– 0.72	– 0.95	– 0.19	– 0.68	– 0.72	– 0.95	**– 1.0**
	3	6.08	5.53	5.85	5.34	5.28	5.05	5.81	5.32	5.28	5.05	**– 5.0**
	4	– 50.92	– 51.47	– 51.15	– 51.66	– 51.72	– 51.95	– 51.19	– 51.68	– 51.72	– 51.95	0.0
$\delta x^{(1)}$ [mm]	–	3.08	2.53	2.85	2.34	2.28	2.05	2.81	2.32	2.28	2.05	**– 2.0**

Notation:
Classic version – classic standardisation (1), criteria k and k_0: see e,g, [13];
Version I – standardisation (5), criterion k and k_0 (11), (12);
Version II – standardisation (5), criterion k and k_0 (20), (21);
Version I*, Version II* – like version I and II (respectively), but classic standardisation (1);
LSQ* – classic *LSQ*, but without deviating observations;
Hampel; QDF – methods of error-insensitive adjustment (see e,g, [13]);
v_i, $\bar{v}_i$, – adjustments from classic LSQ and standardised adjustments (respectively);
δx – increment to the approximate unknown from the classic *LSQ*;
$v_i^{(1)}$, $\delta x^{(1)}$, – similarly to and δx (respectively), but after the first iteration;
The other symbols – like in Fig. 1 and 4.

calculations is shown in the bottom part of the table. The results for the „classic version“ and *LSQ* have been taken (for comparison) from the example mentioned in [2]. Table 2 shows the results of adjustment after the first iteration ($v_i^{(1)}$, $d\ x^{(1)}$). Application of further iterations would have to be followed by subsequent corrections of controlling parameters of a damping function. The aim of the example is to show that it is possible to obtain the correct results (with properly selected parameters of a damping function) in the first iteration.

Judging by the adjustment results, the best outcome is produced by the method adopted for version II. The values of adjustments of observations and of the unknown value are nearly identical to the expected values (LSQ* – a set of observations without gross errors). Of the two applied damping functions, the QDF function [2] is more beneficial. As it turns out, it does not matter much how adjustments are standardised (version II* – classic standardisation (see, e.g. [13]), version II – standardisation proposed in this paper).

4 Summary and conclusions

This paper proposes a new, simplified manner of standardisation of observation adjustments and establishing criteria (controlling parameters) of a damping function, applied in the method of error-insensitive adjustment. The proposed standardisation is directly based on the knowledge of a mean unit error of the observation. The actual controlling parameters (k_0, k) of a damping function are determined based on the proposed auxiliary criterion μ, which is earlier determined from the standardised adjustments. Two alternative methods of determination of coefficient μ have been proposed. The principle has been presented of adopting the restricting values of the coefficient (μ_0, μ_{max}). The theoretical formulae have been illustrated on diagrams, based on example data. The proposed method has been practically verified with a numerical example (exact adjustment of multiple observations of one value). In order to compare the method effectiveness, an adjustment has been performed with the use of the Hampel's function and the *QDF* function [2] where both traditional and the proposed (alternative) methods of standardisation and determination of controlling parameters were applied. The performed tests allow for the following conclusions:

– the proposed, simplified method of standardisation, does not negatively affect the final results of adjustment,
– applying an auxiliary criterion μ allows for determination of better values of controlling parameters (k_0, k) of a damping function than the use of the traditional methods,
– owing to the proposed method of establishing the criteria, the damping function is more effective (compared with other methods): correct observations are not distorted, whereas an observation bearing a gross error is rejected or strongly dampened.

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